EXPERIMENTAL RISK-SENSITIVE OPTIMAL SCHEDULING

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1. Introduction. In this paper we study a problem of scheduling jobs with random processing times under a risk sensitive optimality criterion. We assume that risk sensitivity is given by an exponential disutility function $U_\gamma(x) = (sgn \gamma) \exp(\gamma x)$, $\gamma \in \mathbb{R}$, $\gamma \neq 0$. In Section 2 we present in detail the job scheduling model we consider and present its standard formulation as a Controlled Markov Process (see [7] and [3]). To facilitate comparisons with the results we derive in this paper, in Section 3 we include the analysis of the stochastic optimal control problem corresponding to the risk null performance criterion given by an expected total weighted completion time (see [3] and [7]). We present both a detailed Dynamic Programming (DP) algorithm as well as an interchange argument. In Section 4 we introduce risk-sensitivity by considering the minimization of the expected exponential utility of the total weighted completion time. We develop the corresponding DP algorithm from which the risk-sensitive optimal policies (schedules) are obtained. It is interesting to note, as we shall show, that for the risk-sensitive criterion a simple interchange argument is not applicable, and thus the only general computational and analytical tool for this situation is the DP algorithm that we develop here. Finally, by means of a simple example, we illustrate in Section 5 how the optimal schedule depends on the risk sensitivity coefficient $\gamma$. For extended discussion of the results of this paper see [2].

2. Model Formulation. Following the standard framework presentation, e.g., [1], [3], [4], [9], let us consider a discrete time, stationary, Controlled Markov Process (CMP) specified by $(X, A, P, C)$, where

a) $X$, the state space, is a countable set, say $X = \{1, 2, \ldots\}$. The elements of $X$ are called states.

b) $A$, the action (or control) set, is a finite set. To each $x \in X$ we associate a non-empty subset $A(x)$ of $A$. $A(x)$ represents the set of admissible actions when
the system is in state $x$. The set $K := \{(x, a) : x \in X, a \in A(x)\}$, is called the set of admissible state-action pairs.

c) $P$, the transition kernel, is a family of transition probabilities on $X$ given $K$:

$$P = \{P(\cdot \mid x, a) : (x, a) \in K\}.$$ 

We will also denote $p_{xx'}(a) := P(x' \mid x, a)$.

d) $C : K \to \mathbb{R}$ is the so called one-stage cost function.

The defined above CMP represents a stochastic dynamical system observed at times $t \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The total period of time over which the system is to be observed is called the horizon, and it is denoted by $T$. The evolution of the system is as follows. Let $X_t$ denote the state at time $t$, and $A_t$ the action chosen at that time. If the system is in state $X_t = x \in X$, and the control $A_t = a \in A(x)$ is chosen then: (i) a cost $C(x, a)$ is incurred; and (ii) the system moves to a new state $X_{t+1}$ according to the probability distribution $P(\cdot \mid x, a)$. Once the transition into the new state has occurred, a new action is chosen, and the process is repeated. For some generalizations of this model see [1], [4], [9], Section 2.3.

The key elements of a job scheduling problem are as follows; see [3] and [7].

- $N$: number of jobs to process.
- $T_i$: processing time of job $i$. We assume that the processing times are independent random variables with known distribution.
- $w_i$: weight (usually interpreted as a holding cost) of job $i$.
- $Z_i$: random completion time of job $i$. The distribution of the completion times depend on the chosen schedule.

The problem faced is as follows. We have the $N$ jobs with independent processing times that are to be processed nonpreemptively (that is, determined before any
processing begins) on a single machine. For simplicity we assume there are no arrivals, breakdowns, setup or switchover costs, or precedences. If job \( j \) is completed at time \( t \), the cost incurred is \( w_j t \), where \( w_j > 0 \). The problem is to find a job schedule such that the expected total weighted completion time \( E \left[ \sum_j w_j Z_j \right] \) is minimized, where the expectation is understood to be taken with respect to the joint distribution of the \( \{ T_i \} \).

A formulation of the previous problem within the context of CMP's is as follows. Define \( x_0 := \phi \) (the empty set), \( x_t := \{ \text{the collection of jobs completed up to time } t \} \). The (finite) state space \( X \) is defined as the collection of subsets of \( \{1, 2, \ldots, N\} \), i.e., \( X = 2^{\{1,2,\ldots,N\}} \), and \( A := \{1, 2, \ldots, N\} \) denotes the action space. The set of state-admissible action pairs is given by \( K = \{(i, a) : i \in X, a \in i^c\} \). The model is event driven, i.e., time is incremented when a job is completed. Let \( a_t \) denote the action taken at time \( t \), i.e., the \( t \)-th scheduled job, and compute

\[
Y_0 = T_{a_0}, \quad Y_t = Y_{t-1} + T_{a_t}, \quad t = 1, 2, \ldots, N - 1.
\]

(1)

Let the one stage cost function be given as

\[
\tilde{C}(x_t, a_t, Y_t) = w_{a_t} Y_t = w_{a_t} \left( \sum_{i \in x_t} T_i + T_{a_t} \right)
\]

(2)

Then, the CMP formulation of the problem above is obtained by computing the expected one-stage cost

\[
C(x_t, a_t) = w_{a_t} \left( \sum_{i \in x_t} E T_i + E T_{a_t} \right),
\]

(3)

and the transition kernel \( P \) as

\[
P(j | i, a) = \begin{cases} 1 & \text{if } j = i \cup \{a\} \\ 0 & \text{otherwise,} \end{cases}
\]

for \( i, j \in X \). Thus the MDM is given by the four-tuple \( (X, A, P, C) \).

**Remark 1** Note that given the linear dependence of risk neutral criteria on the one stage costs, all that one needs in our case is the mean processing times given by (3). However, as we shall see later, for risk-sensitive criteria the dependence of the one stage cost on the "disturbance" \( T_{a_t} \) will need to be made explicit, and hence the mean processing times will not suffice in that situation. Thus, the explicit use of (2) will be needed.

An admissible Markovian deterministic policy \( \pi = \{f_0, f_1, \ldots, f_{N-1}\} \in \Pi_{MD} \) satisfies \( f_t(i) \in i^c \). To the deterministic policy \( \pi = \{f_0, \ldots, f_{N-1}\} \) corresponds the sequential order of the jobs \( ( f_0(\phi) , f_1(\{f_0(\phi)\}) , f_2(\{f_0(\phi) , f_1(\{f_0(\phi)\})\}) , \ldots ) \). Therefore, a policy \( \pi \in \Pi_{MD} \) can be associated with an open-loop schedule \( \{a_0, \ldots, a_{N-1}\} \in A^N \).

Given that the initial state is always \( \phi \), each policy \( \pi \) induces a unique probability \( P^\pi \) on the sample space \( \Omega = (X \times A)^{N-1} \times X \). The expected value with respect to \( P^\pi \) is denoted by \( E^\pi \).

**3. Risk Neutral Case.** Let \( \pi = (a_0, a_1, \ldots, a_{N-1}) \in \Pi_{MD} \). The total cost incurred by \( \pi \) is given by

\[
J_N^\pi(\phi) = C(\phi, a_0) + \ldots + C(\{a_0, \ldots, a_{N-2}\}, a_{N-1}).
\]

(4)

The optimal cost, \( J_N^*(\phi) := \inf \{ J_N^\pi(\phi) \} \), satisfies the last step of the standard Dynamic Programming recursion:

\[
J_0(i) = 0,
\]

\[
J_t(i) = \min_{a \in \Gamma} \left\{ w_a \sum_{k \in i \cup \{a\}} E[T_k] + J_{t-1}(i \cup \{a\}) \right\},
\]

(5)

Of course, in this case the optimal schedule could also be obtained by total enumeration. As an illustration of this algorithm, we obtain next, the optimal schedule, and the optimal cost for a simple problem with 3 jobs.

**Example 1.** We suppose that there are 3 jobs to be processed in sequential order. Let \( f_1, f_2, f_3 \) be the distributions of \( T_1, T_2, T_3 \) respectively, where \( f_1, f_3 \) are concentrated in 1, and \( f_2 \) is given by

\[
f_2 = \begin{cases} 0 & \text{with probability } 9/20 \\ 1 & \text{with probability } 3/5 \\ 10 & \text{with probability } 1/20 \end{cases}
\]

(5)

The weights and expected processing times are given by

\[
\begin{align*}
\text{jobs} & \quad 1 & \quad 2 & \quad 3 \\
w_j & \quad 6 & \quad 10 & \quad 8 \\
E(T_j) & \quad 1 & \quad 1 & \quad 1
\end{align*}
\]

We have that \( J_0((1,2,3)) = 0 \),

\[
J_1((1,2)) = \frac{w_3 E[T_1 + T_2 + T_3]}{24}, \quad J_1((1,3)) = \frac{w_2 E[T_1 + T_2 + T_3]}{30}, \quad J_1((2,3)) = \frac{w_1 E[T_1 + T_2 + T_3]}{18}.
\]

Substituting these values in \( J_2 \) we obtain

\[
J_2((1)) = \min \{ w_2 E[T_1 + T_2] + J_1((1,2)), w_3 E[T_1 + T_3] + J_1((1,3)) \} = 44,
\]

\[
J_2((2)) = \min \{ w_1 E[T_1 + T_2] + J_1((1,2)), w_3 E[T_2 + T_3] + J_1((2,3)) \} = 34,
\]

\[
J_2((3)) = \min \{ w_1 E[T_1 + T_3] + J_1((1,3)), w_2 E[T_2 + T_3] + J_1((2,3)) \} = 38.
\]
Finally, we calculate the optimal cost:

\[
J_N^*(\phi) = \min \{ w_1 E(T_1) + J_2(\{1\}) , w_2 E(T_2) + J_3(\{2\}) \} \\
= \min \{ 6 + 44,10 + 34,8 + 38 \} = 44,
\]

and the optimal schedule is \( S = (231) \).

**Interchange Argument.** In problems for which there exists an optimal open loop policy, an interchange argument may be used to obtain (necessary) optimality conditions [3], [7]. Let

\[
\pi^* = (i_0^*, i_1^*, \ldots, i_{r-1}^*, i_j^*, i_{r+2}^*, \ldots, i_N^*)
\]

be an optimal schedule, and consider the schedule \( \pi' = (i_0^*, \ldots, i_{r-1}^*, i_j, i_{r+2}^*, \ldots, i_N^*) \) obtained from \( \pi^* \) by interchanging the jobs \( i \) and \( j \). We have that

\[
J_N^*(\phi) = c_1 + C((i_0^*, \ldots, i_{r-1}^*, i_j), i) + c_2,
\]

and

\[
J_N^*(\phi) = c_1 + C((i_0^*, \ldots, i_{r-1}^*, j), i) + c_2,
\]

where

\[
c_1 = C(\phi, i_0^*) + C((i_0^*, i_1^*), i_1^*) + \ldots + C((i_0^*, \ldots, i_{r-2}^*), i_{r-1}^*),
\]

and

\[
c_2 = C((i_0^*, \ldots, i_j), i_{r+2}^*) + \ldots + C((i_0^*, \ldots, i_{N-1}^*), i_N^*).
\]

Since \( \pi^* \) is supposed to be optimal, we obtain

\[
w_1 E(T_r + T_i) + w_2 E(T_r + T_j + T_i) \leq w_1 E(T_r + T_i) + w_1 E(T_r + T_j + T_i),
\]

where \( t_r = \sum_{k=0}^{r-1} T_i^* \). Cancelling equal terms in the last inequality we obtain

\[
w_1 \frac{E(T_i)}{E(T_j)} \leq w_1 \frac{E(T_j)}{E(T_i)}.
\]

**Proposition 1** The rule that process the jobs in decreasing order of \( \frac{E(T_i)}{E(T_j)} \) is optimal.

**Remark 2** In the optimal schedule we have that job 2, the first job scheduled, is the one with processing time having largest variance. Note that the variability of job 2 impacts greatly the variability of the total cost. For example if \( T_2 \) takes the value 10, then the total cost will be 10(10) + 5(10 + 1) + 6(10 + 1 + 1) = 260, while if \( T_2 \) takes the value 0, then the total cost will be 20. Thus a risk averse Decision Maker (DM) may prefer to schedule later the job 2; that is, the schedule \( S = (231) \) may not to be his/her optimal schedule.

4. Risk Sensitive Case. Suppose now that the Decision Maker (DM) has constant risk sensitivity coefficient \( \gamma \neq 0 \) [5], [10] and [8]. The case \( \gamma > 0 \) corresponds to a risk-averse (risk-seeking) DM, and if \( \gamma = 0 \) we recover the standard risk-neutral situation. Thus the exponential total cost due to a policy \( \pi = (a_0, a_1, \ldots, a_{N-1}) \) would be computed as

\[
J_N^*(\phi, \gamma) = E^\gamma \left[ \sum_{t=0}^{N-1} \bar{C}(x_t, a_t, Y_t) \right]
\]

and thus one sees that (3) is not enough to compute (7), but instead (2) is needed. Hence the modeling framework needs to be modified to be able to take explicitly into consideration the dependence of the one-stage cost on a random disturbance \( Y_t \). (See also [3]). We do this as follows. The random disturbance \( Y_t \) given by (1) takes values \( \nu_t \in R^+ \) and has a distribution

\[
P(Y_0 \in B) = P_{T_{a_0}}(B),
\]

\[
P(Y_t \in B | \nu_{t-1}, a_t) = P_{r_{t-1} + T_{a_t}}(B), \quad t = 1, \ldots, N-1,
\]

where \( P_X \) denotes the distribution of the random variable \( X \). The modified sample space is given by \( \Omega := (X \times A \times R)^{N-1} \times X \). The history spaces are given by \( \mathcal{H}_0 = \emptyset \), \( \mathcal{H}_t = (X \times A \times R)^{t-1} \times X \). For each \( (\nu, a) \in R \times A \), define a probability kernel on \( R \) by

\[
Q(\cdot | \nu, a) := P_{\nu + T_a}(\cdot).
\]

Let \( Q := \{ Q(\cdot | \nu, a) : (\nu, a) \in R \times A \} \). Thus, the basic elements to study this scheduling problem with risk sensitive criteria are \( (X, A, P, Q, C) \). For \( \pi \in \Pi \) the set of all admissible policies, \( \tilde{P}_\pi \) and \( \tilde{E}_\pi \) denote the appropriate operators. \( X_t, A_t, Y_t \) are defined as the projection maps on \( (\Omega, \mathcal{B}(\Omega)) \). Let \( \pi = (q_0, \ldots, q_{N-1}) \in \Pi \). For \( t < N \) define \( \tilde{J}_t \) as:

\[
\tilde{J}_t(h_t, \gamma) = \tilde{E}_h \left[ \sum_{t=1}^{N-1} \bar{C}(X_t, A_t, Y_t) \right]
\]

and for \( t = N \) by \( J_N^*(x_N) = sgn \gamma \). Let \( \tilde{J}_t(h_t, \gamma) := \inf_{\pi \in \Pi} \tilde{J}_t^*(h_t, \gamma) \). It denotes the infimum, over all policies, of the expected exponential cost from decision epoch \( t+1 \) onward when the history up to time \( t \) is \( h_t \). The following proposition shows that \( J_0(\phi, \gamma) \) satisfies the last step of a backward recursion.

**Theorem 2 (Dynamic Programming Algorithm)** The optimal exponential total cost \( J_0(\phi, \gamma) \) satisfies the last step of the recursion:

\[
\tilde{J}_N(\{1, 2, \ldots, N\}) = sgn \gamma
\]

and

\[
	ilde{J}_t(h_t, \gamma) = \min_{a_t} \left\{ \int Q(\nu_t | \nu_{t-1}, a_t) \right\}
\]

\[
\left\{ e^{\gamma C(x_t, a_t, \nu_t)} \tilde{J}_{t+1}(h_t, a_t, \nu_t, x_{t+1}) \right\},
\]
where \( x_t = \{ a_0, \ldots, a_{t-1} \} \), and \( h_t = (\phi, a_0, \nu_0, \ldots, x_t) \).

**Proof:** Let \( \pi = (q_0, \ldots, q_{N-1}) \in \Pi \). We have that

\[
\hat{\bar{J}}_1^n(h_t, \gamma) \geq \sum_{a_t} q_t(a_t | h_t) \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, a_t)
\left[ e^{\gamma C(x_t, a_t, \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, a_t, \nu_t, x_{t+1}) \right]
\geq \sum_{a_t} q_t(a_t | h_t) \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, a_t)
\left[ e^{\gamma C(x_t, a_t, \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, a_t, \nu_t, x_{t+1}) \right]
\geq \sum_{a_t} q_t(a_t | h_t) \min_{a_t} \left\{ \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, a_t)
\left[ e^{\gamma C(x_t, a_t, \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, a_t, \nu_t, x_{t+1}) \right]\right\}
\geq \min_{a_t} \left\{ \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, a_t)
\left[ e^{\gamma C(x_t, a_t, \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, a_t, \nu_t, x_{t+1}) \right]\right\}.
\]

To see the reverse inequality, consider the function \( f : H_t \rightarrow \mathbb{A} \) such that \( f(h_t) \) is determined by

\[
\int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, f(h_t))
\left[ e^{\gamma C(x_t, f(h_t), \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, f(h_t), \nu_t, x_{t+1}) \right]
= \min_{a_t} \left\{ \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, a_t)
\left[ e^{\gamma C(x_t, a_t, \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, a_t, \nu_t) \right]\right\}.
\]

Consider the policy \( \pi' \) that chooses \( f(h_t) \) at time \( t \), and if the next history is \( h_{t+1} = (h_t, f(h_t), \nu_t, x_{t+1}) \) then it uses the policy \( \pi(\nu) \) such that

\[
\hat{J}^{\pi'}_{t+1}(h_t, \gamma) \leq \hat{J}_{t+1}(h_t, \gamma) + \epsilon.
\]

Thus, we have that

\[
\hat{J}_1(h_t, \gamma) \leq \hat{J}^{\pi'}_{t}(h_t, \gamma)
\leq \sum_{a_t} q_t(a_t | h_t) \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, f(h_t))
\left[ e^{\gamma C(x_t, f(h_t), \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, f(h_t), \nu_t) \right]
+ \epsilon \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, f(h_t)) \left[ e^{\gamma C(x_t, f(h_t), \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, f(h_t), \nu_t) \right]
= \min_{a_t} \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, a_t) e^{\gamma C(x_t, a_t, \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, a_t, \nu_t, x_{t+1})
+ \epsilon \int_{\mathbb{R}} Q(d\nu_t | \nu_{t-1}, f(h_t)) \left[ e^{\gamma C(x_t, f(h_t), \nu_t)} \hat{J}^{\pi}_{t+1}(h_t, f(h_t), \nu_t) \right].
\]

Since \( \epsilon \) was arbitrary the result of the theorem follows.

**Remark 3** Notice that \( \hat{J}_1(h_t, \gamma) \), where \( h_t = (\phi, a_0, \nu_0, x_1, \ldots, \nu_{t-1}, x_t) \), does not depend on the whole history but only on \( \nu_{t-1} \) and \( x_t \).
Thus from (8), (9) and (10), we obtain

\[
\tilde{J}_0(\phi, \gamma) = \int_{\mathbb{R}} Q(du_0 | v_0, 1)e^{u_0v_0} \int_{\mathbb{R}} Q(du_1 | v_0, 1)e^{u_1v_1} \int_{\mathbb{R}} Q(du_2 | v_1)e^{u_2v_2} = E[e^{w_1 + w_2 + w_3}T_3]E[e^{w_1 + w_2}T_1]E[e^{w_2}T_2]
\] (11)

Thus from (11) it follows that the optimal schedule is \( S = (3 \, 1 \, 2) \), which, as we expected (see Remark 2), is different to the optimal schedule \( S = (2 \, 3 \, 1) \) obtained in the risk neutral case, and one that schedules the most uncertain job for last. Now, since the optimal schedule is an open-loop policy and motivated by the risk neutral case, we can pose the following question.

**Question:** Applying an interchange argument, can we derive some simple necessary and sufficient conditions for optimality?

We proceed to investigate this question. Suppose that the schedule \( \pi^* = (i_0, \ldots, i_{r-1}, j, i_r, i_{r+1}, \ldots, i_{N-1}) \) is optimal. Let \( M_l = \sum_{k=l}^{N-r-1} w_{ik} \), for \( l = 0, 1, \ldots, r-1 \), and let \( S_{r+l} = \sum_{k=r-l}^{N-l} w_{ik} \), for \( l = 2, 3, \ldots, N-r-1 \). We have that

\[
\tilde{J}_0^*(\phi, \gamma) = \text{sgn}\gamma K_1 E[e^{\gamma(w_1 + w_2 + S_{r+2})T_2}] \cdot E[e^{\gamma(w_1 + S_{r+2})T_1}] K_2,
\]

where \( K_2 = E[e^{\gamma S_{r+2}T_{r+2}}] \ldots E[e^{\gamma S_{N-1}T_{N-1}}] \), and

\[
K_1 = E[e^{\gamma(M_0 + w_1 + w_2 + S_{r+3})T_0}] \ldots E[e^{\gamma(M_{r-1} + w_1 + w_2 + S_{r+3})T_{r-1}}],
\]

Let \( \pi' = (i_0, \ldots, i_{r-1}, j, i_r, i_{r+2}, \ldots, i_{N-1}) \) be the schedule which interchanges the jobs \( i \) and \( j \). Then

\[
\tilde{J}_0^*(\phi, \gamma) = \text{sgn}\gamma K_1 E[e^{\gamma(w_1 + w_2 + S_{r+3})T_1}] \cdot E[e^{\gamma(w_1 + S_{r+2})T_2}] K_2.
\]

Since the policy \( \pi^* \) is supposed to be optimal, we have that, if \( \gamma > 0 \) then

\[
E[e^{\gamma(w_1 + w_2 + S_{r+3})T_1}] E[e^{\gamma(w_1 + S_{r+2})T_2}] \leq E[e^{\gamma(w_1 + w_2 + S_{r+3})T_2}] E[e^{\gamma(w_1 + S_{r+2})T_1}],
\]

and if \( \gamma < 0 \) then the reverse inequality holds. Hence, we arrive at the following answer.

**Answer:** As opposed to the risk-neutral case, we notice that by applying an interchange argument, we do not derive any simple general condition for optimality. Observe that (12) involves the jobs scheduled from the job \( i \) onward. Thus, in this case, the Dynamic Programming Algorithm is the only way we have to compute the optimal schedule. Certainly, a possibility is to explore particular distributions for \( \{ T_i \} \), e.g., see [6], and perhaps (12) may then yield simpler optimality conditions.

**5. Optimal Schedule Dependence on \( \gamma \)** In this section, we will illustrate, via a simple example with 2 jobs, how the optimal schedule depends on \( \gamma \). We will show that there exists \( \gamma^* > 0 \) such that \( S = (1 \, 2) \) will be optimal for a DM with risk coefficient less or equal than \( \gamma^* \), while \( S = (2 \, 1) \) will be optimal for a DM with risk coefficient great or equal than \( \gamma^* \).

**Example 2.** Suppose that we want to process 2 jobs whose weights and processing times are given by

<table>
<thead>
<tr>
<th>jobs</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_j )</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

\( T_1 \) takes value 2 and 4 with probability \( \frac{1}{2} \) each one, and \( T_2 \) is concentrate in 3. Note that \( E(T_1) = E(T_2) = 3 \). Thus

\[
\frac{w_1}{E(T_1)} > \frac{w_2}{E(T_2)},
\]

from Proposition 1 we have that

\[
S = (1 \, 2) \quad \text{is optimal if DM is risk-neutral.}
\]

We consider now a risk-sensitive DM. Note that in any example with 2 jobs, the inequalities (12) turn out to be very simple to manipulate. We have that

\[
E[e^{\gamma(w_1 + w_2)T_1}] = \frac{e^{2\gamma} + e^{5\gamma}}{2}
\]

\[
E[e^{\gamma w_2 T_2}] = e^{18\gamma}
\]

\[
E[e^{\gamma(w_1 + w_2)T_2}] = e^{30\gamma}
\]

\[
E[e^{\gamma w_1 T_1}] = \frac{e^{14\gamma} + e^{28\gamma}}{2}.
\]

Thus,

\[
2 \left( E[e^{\gamma(w_1 + w_2)T_1}] E[e^{\gamma w_2 T_2}] - E[e^{\gamma(w_1 + w_2)T_2}] E[e^{\gamma w_1 T_1}] \right) = \left( e^{26\gamma} + e^{52\gamma} \right) e^{18\gamma} - e^{30\gamma} \left( e^{14\gamma} + e^{28\gamma} \right) = e^{44\gamma} \left( e^{26\gamma} - e^{23\gamma} - e^{9\gamma} + 1 \right).
\] (13)

**Lemma 3** Let \( g(\gamma) := e^{26\gamma} - e^{23\gamma} - e^{9\gamma} + 1 \). Then we have the following.

i) \( g(\gamma) \) has only two real zeros:

\[
\gamma_1 = 0, \quad \text{and} \quad \gamma^* \in (0, 1).
\]

ii) \( g(\gamma) < 0 \) if \( 0 < \gamma < \gamma^* \).

iii) \( g(\gamma) > 0 \) if \( \gamma < 0 \), or \( \gamma > \gamma^* \).
Then from Lemma 3 and (13) we will obtain
\[ E[e^{T(w_1+w_2)T_1}|E[e^{T_{w_2}T_2}] > E[e^{T(w_1+w_2)T_3}|E[e^{w_3T_1}], \]
if \( \gamma < 0 \), or \( \gamma > \gamma^* \), and the reverse inequality if \( 0 < \gamma < \gamma^* \), and hence
\[ S = (1 2) \text{ is optimal for a DM with risk coefficient less than or equal to } \gamma^*, \]
\[ S = (2 1) \text{ is optimal for a DM with risk coefficient greater than or equal to } \gamma. \]

**Proof:** (of Lemma 3) We have that
\[ g'(\gamma) = 26e^{2\gamma} - 23e^{2\gamma} - 9e^{9\gamma}, \]
\[ = e^{9\gamma}h(\gamma), \quad (14) \]
where
\[ h(\gamma) = 26e^{12\gamma} - 23e^{14\gamma} - 9. \]
Let \( \gamma < 0 \), and \( b = -\gamma \). Then
\[ g'(\gamma) = g'(-b) = e^{-9b}(26e^{-17b} - 23e^{-14b} - 9) \]
\[ = e^{-9\gamma} \left( \frac{26 - 23e^{3b} - 9e^{17b}}{e^{17b}} \right) < 0. \]
Now, \( g' \) has a zero in \((0,1)\) since \( g'(0) = 26 - 23 - 9 < 0 \), and \( g'(1) > 0 \). It is easy to see that \( h'(\gamma) = 0 \) if and only if \( \gamma = \frac{1}{3} \ln \frac{19}{22} \). Therefore \( h(\gamma) \) has at most two real zeros and hence, from (14), \( g' \) has at most two real zeros. Let \( \gamma > 0 \) be the smallest zero (if there exist two) of \( g' \). Then since \( g'(0) = -6 \) it holds that \( g'(\gamma) < 0 \), for all \( \gamma \in (0,\gamma) \), and therefore \( g \) is decreasing in \((0,\gamma) \).
Since \( g(0) = 0 \) we have that
\[ g(\gamma) < 0, \text{ for all } \gamma \in (0,\gamma). \quad (15) \]
Now, \( g(1) = e^9(e^{17} - e^{14} - 1) > 0 \). Therefore there exists \( \gamma^* \in (\gamma, 1) \) such that \( g(\gamma^*) = 0 \), and
\[ g(\gamma) < 0, \text{ for all } \gamma \in (\gamma, \gamma^*). \quad (16) \]
Thus (14) and (16) imply ii). Now, since for \( \gamma < 0 \), \( g'(\gamma) < 0 \), we have that \( g \) is decreasing in \((-\infty, 0) \), and since \( g(0) = 0 \), we have that
\[ g(\gamma) > 0, \text{ for all } \gamma \in (-\infty, 0). \quad (17) \]
Finally,
\[ g(\gamma) > 0, \text{ if } \gamma > 1. \quad (18) \]
Thus, (17) and (18) imply iii). Now,
\[ g'(\gamma) = 0 \iff \gamma = \frac{\ln \frac{3}{2}}{3}. \quad (19) \]
Therefore \( g \) has at most two real zeros. One of them is clearly \( \gamma_1 = 0 \), and the other one (if it exists) is \( \gamma^* > \frac{\ln \frac{3}{2}}{3} \). Moreover,
\[ \gamma < \frac{\ln \frac{3}{2}}{3} \Rightarrow g \text{ is decreasing on } (-\infty, \frac{\ln \frac{3}{2}}{3}). \quad (20) \]
\[ \gamma > \frac{\ln \frac{3}{2}}{3} \Rightarrow g \text{ is increasing on } (\frac{\ln \frac{3}{2}}{3}, \infty). \quad (21) \]
i), ii), and iii) follow from (19), (20), and (21), since \( \gamma_1 = 0 \) is a zero of \( g \).