Reduced-order robust adaptive control design of uncertain
SISO linear systems

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SUMMARY

In this paper, a stability and robustness preserving adaptive controller order-reduction method is developed
for a class of uncertain linear systems affected by system and measurement noises. In this method, we
immediately start the integrator backstepping procedure of the controller design without first stabilizing a
filtered dynamics of the output. This relieves us from generating the reference trajectory for the filtered
dynamics of the output and thus reducing the controller order by \( n \), \( n \) being the dimension of the
system state. The stability of the filtered dynamics is indirectly proved \( \text{via} \) an existing state signal. The
trade-off for this order reduction is that the worst-case estimate for the expanded state vector has to be
chosen as a suboptimal choice rather than the optimal choice. It is shown that the resulting reduced-order
adaptive controller preserves the stability and robustness properties of the full-order adaptive controller
in disturbance attenuation, boundedness of closed-loop signals, and output tracking. The proposed order-
reduction scheme is also applied to a class of single-input single-output linear systems with partly measured
disturbances. Two examples are presented to illustrate the performance of the reduced-order controller in
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1. INTRODUCTION

Adaptive control has been an important research topic in control theory. Based on the approach
adopted to address the problem, adaptive control design may be classified into three categories:
certainty equivalence (CE)-based design, nonlinear adaptive control design, and worst-case analysis
(WCA)-based design.

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Early development of adaptive control since 1970s has been dominated by the CE principle [1, 2], which decouples the parameter estimation design from the control design by making use of some standard parameter estimators and supplying the estimates to control law as if they were true parameters. The CE-based design simplifies the controller structure considerably and leads to many successful applications [3–5] for linear systems. On the other hand, early designs using this approach are shown to be nonrobust when the system has unmodeled dynamics and (or) deterministic exogenous disturbance inputs [6, 7], which motivates the study of robust adaptive control design in the 1980s and 1990s, where the CE-based designs are modified to render the closed-loop system robust. These results still fell short of directly addressing the disturbance attenuation property for the adaptive system. Furthermore, the CE-based approach is unsuccessful in generalizing this approach to the nonlinear systems with severe nonlinearity, which motivates the study of nonlinear adaptive control design in the 1990s.

For nonlinear adaptive control design, one of its major research focus is in (partially) feedback linearizable systems that are geometrically characterized in [8]. The introduction of recursive integrator backstepping methodology [9] provides a systematic design tool to obtain the adaptive control law for the class of parametric strict- (or pure-) feedback nonlinear systems. This method admits great design flexibility evident in the selection of the value function and the virtual control laws. See the book [10] for a complete list of references. The integrator backstepping methodology has also been applied to study adaptive control for parametric strict-feedback nonlinear systems with unknown sign of high-frequency gain [11].

In addition to the literature we mentioned above, plenty of methodologies of linear and nonlinear adaptive control design have been developed to deal with different control problems. Chapter 6 of [12] presents various design methodologies for adaptive control of different classes of uncertain systems subject to external disturbances. Adaptive control algorithms with the implicit reference model are discussed in [12, 13]. Additionally, the adaptive control problem for systems with incomplete measurements has been given attention in [14–18]. In [19], the problem of adaptive compensation of unknown external disturbances has been solved for a class of nonlinear systems with globally designed normal norm. These works are inspiring for later research on the adaptive control of the uncertain systems with external disturbances.

Another category of adaptive control design approach is the WCA-based adaptive control. This approach has been motivated by the success of $H^{\infty}$-optimal control, which provides a solution to the robust control problem by studying only the disturbance attenuation property of the closed-loop system. The game-theoretic methodology to $H^{\infty}$-optimal control problem offers the most promising way to address nonlinear $H^{\infty}$-optimal control problems [20] with a lot of success [21–25]. The WCA-based adaptive control formulates a robust adaptive control problem as a nonlinear $H^{\infty}$-optimal control problem under imperfect state measurements, by capturing the objectives of robust adaptive control with a single game-theoretic cost function. In this approach, the unknown parameters are treated as part of the expanded state vector. An application of the cost-to-come function analysis [22, 26] to the nonlinear $H^{\infty}$-optimal control problem yields a finite dimensional estimator for the expanded system and converts the $H^{\infty}$-optimal control problem with imperfect state measurements into one with full-information measurements. Then, the integrator backstepping methodology is applied to solve this full-information measurement problem. The above design paradigm has been successfully applied to identify problems [27, 28] and robust adaptive control problems [29–33], which indicate that the resulted identifiers and adaptive controllers have strong
robustness. With a practical consideration that lower-order controllers are easy to understand, maintain, and implement in the real world, reduced-order adaptive controller design is studied in [34], where the controller order is reduced by $n - 1$ or $n - 2$ depending on the eigenstructure of a feedback matrix, as compared with the full-order controller design proposed in [32]. It is proved that the closed-loop system, after order reduction, achieves the same strong robustness properties as [32]. Encouraged by these successes, research goes further on this topic.

In this paper, we continue to study the reduced-order adaptive control design in the framework of the worst-case-based approach. An order-reduction methodology is presented to reduce the controller structures in [32, 34], while preserving the stability and robustness properties of these controllers. In this paper, the system under consideration is a class of uncertain single-input single-output (SISO) linear systems that are subject to measurement noises and exogenous disturbances. We assume the system is observable, admits a transfer function that is strictly minimum phase (SMP) with known relative degree. The true system may be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and the uncontrollable modes on the imaginary axis are uncontrollable by the disturbance input. Then, we apply the WCA-based formula as discussed in the fifth paragraph to deal with this robust adaptive control problem. Two steps are involved in the design procedure. One is the estimation design step and the other is the controller design step. The main difference between the reduced-order control design investigated in this paper and the full-order adaptive control design starts in the controller design step. As the full-order controller design, we apply the integrator backstepping methodology to derive the control law. However, the controller design begins from step 1, without first stabilizing the filtered dynamics of the output as step 0 does in the full-order controller design. This relieves us from generating the reference trajectory for state of the filtered dynamics of the output to track. Therefore, the controller structure can be simplified by $n$ integrators, where $n$ is the upper bound of the order of the unknown system.

The lack of step 0 results in the lack of one nonpositive drift term in the closed-loop value function, which may degrade the transient performance of the reduced-order controller. In addition, in order to guarantee the boundedness of the closed-loop signals, the worst-case estimate for the expanded state will be chosen sub-optimally rather than optimally. Exactly the same robustness results are established for the reduced-order controller as those of the full-order controller. The results encompass the following three aspects: boundedness of all closed-loop signals, desired disturbance attenuation property, and asymptotic tracking property. In this paper, we extend the proposed order-reduction scheme to simplify the full-order control design for a class of SISO linear systems with partly measured disturbances [35]. Moreover, this order-reduction scheme can also be extended to a class of SIMO linear systems [36] and a special class of multi-input multi-output (MIMO) linear systems [37].

The organization of this paper is as follows. In Section 2, we list the notations to be used in this paper. We present the problem formulation of the robust adaptive control problem in Section 3. Then, we present a summary of estimation design in Section 4 for the convenience of readers. In Section 5, the controller design is discussed in detail, where we make use of the integrator backstepping methodology. In Section 6, we present the main robustness results in the form of a theorem. Then, a numerical example is included in Section 7 to illustrate the performance of the reduced-order controller. We also extend the proposed order-reduction control methodology to a class of SISO linear systems with partly measured disturbances in Section 8. This paper ends with some concluding remarks in Section 9 and two appendices.
2. NOTATIONS

We denote the real line by \( \mathbb{R} \) and the set of natural numbers by \( \mathbb{N} \). We say that a function \( f \) belongs to \( \mathcal{C} \) if it is continuous; we say that it belongs to \( \mathcal{C}_k \) if it is continuously (partial) differentiable up to \( k \)th order. For a vector or matrix \( A, A' \) denotes its transpose. For any \( z \in \mathbb{R}^n \) and any \( n \times n \)-dimensional symmetric matrix \( M, n \in \mathbb{N} \), \( \|z\|^2_M \) denotes \( z'Mz \) and \( |z|^2 \) denotes \( z'z \).

For any \( b \in \mathbb{R} \),
\[
\text{sgn}(b) = \begin{cases} 
-1 & b < 0 \\
0 & b = 0 \\
1 & b > 0 
\end{cases}
\]

For any matrix \( M \), the vector \( \overrightarrow{M} \) is formed by stacking up its column vectors. For any symmetric matrix \( M \), the vector \( \overrightarrow{M} \) is formed by stacking up the column vectors of the lower triangular part of \( M \). For any \( n_1, n_2 \in \mathbb{N} \cup \{0\} \), \( 0_{n_1 \times n_2} \) denotes the \( n_1 \times n_2 \)-dimensional matrix with all zero elements. For \( n \times n \)-dimensional symmetric matrices \( M_1 \) and \( M_2 \), where \( n \in \mathbb{N} \), we express \( M_1 > M_2 \) if \( M_1 - M_2 \) is positive definite; we express \( M_1 \geq M_2 \) if \( M_1 - M_2 \) is positive semi-definite. For any matrix \( M, \|M\|_p \) denotes its \( p \)-induced norm, \( 1 \leq p \leq \infty \). \( L^\infty \) denotes the set of square integrable functions, and \( L^\infty \) denotes the set of bounded functions. For any time function \( y, y|_{[t_0, t_1]} \) denotes the waveform of \( y \) on the interval \([t_0, t_1]\). For \( n \in \mathbb{N} \), \( I_n \) denotes the \( n \times n \)-dimensional identity matrix.

3. PROBLEM FORMULATION

We consider the following uncertain LTI system:
\[
\begin{align*}
\dot{x} &= \hat{A}x + \hat{B}u + \hat{D}w, \quad x(0) = \hat{x}_0 \quad (1a) \\
y &= \hat{C}x + \hat{E}w \quad (1b)
\end{align*}
\]

where \( \dot{x} \) is the \( n \)-dimensional state vector; \( n \in \mathbb{N} \); \( u \) is the scalar control input; \( y \) is the scalar system output; \( \dot{w} \) is the \( q \)-dimensional disturbance input; \( q \in \mathbb{N} \); all signals in the system are assumed to be continuous, i.e. in the space \( \mathcal{C} \); and the constant matrices \( \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E} \) are generally unknown or partially unknown. The true system (1) satisfies the following assumption.

Assumption 1

The pair \( (\hat{A}, \hat{C}) \) is observable. The transfer function \( H(s) = \hat{C}\left(sI_n - \hat{A}\right)^{-1}\hat{B} \) is known to have relative degree \( r \in \mathbb{N} \) and is SMP (here SMP is defined as that the transfer function \( H(s) \) has its zeros in the left-half \( s \)-plane). The uncontrollable part (with respect to the control input \( u \)) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to an eigenvalue of the matrix \( \hat{A} \) on the \( jw \)-axis is uncontrollable from the disturbance \( \dot{w} \).

By Assumption 1, there always exist a state transformation \( x = \hat{T}\dot{x} \) and a disturbance transformation \( w = \hat{M}\dot{w} \), such that the system can be expressed as
\[
\begin{align*}
\dot{x} &= Ax + \left(y\hat{A}_{211} + u\hat{A}_{212}\right)\theta + Bu + Dw, \quad x(0) = x_0 \quad (2a) \\
y &= Cx + Ew \quad (2b)
\end{align*}
\]
where $\tilde{T}$ is an unknown real invertible matrix; $\hat{M}$ is an unknown real $q \times \hat{q}$-dimensional matrix, $q \in \mathbb{N}$; $\theta \in \mathbb{R}^\sigma$ is the vector of unknown constant parameters of the system, $\sigma \in \mathbb{N}$; and the matrices $A, \hat{A}_{211}, \hat{A}_{212}, B, D, C,$ and $E$ are known and admit the following structure:

$$A = \begin{bmatrix}
  a_{11} & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
  a_{21} & a_{22} & 1 & \ldots & 0 & 0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{r-1} & \ldots & \ldots & 1 & 0 & 0 & \ldots & 0 \\
  a_{r-1} & \ldots & \ldots & a_{r-1} & 1 & \ldots & 0 & \vdots \\
  a_{r1} & \ldots & \ldots & a_{rr-1} & a_{rr} & \ldots & a_{r1} & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{n1} & \ldots & \ldots & a_{nr-1} & a_{nr} & \ldots & a_{nn}
\end{bmatrix}, \quad \hat{A}_{212} = \begin{bmatrix}
  0_{(r-1) \times \sigma} \\
  \hat{A}_{2120} \\
  \hat{A}_{212r}
\end{bmatrix}$$

$\hat{A}_{2120}$ is a row vector; $b_{pi}, i = 1, \ldots, n-r$, are constants. Denote the elements of $x$ by $[x_1 \ldots x_n]'$.

Equation (2) is called the design model. Because of the structures of $A, \hat{A}_{212},$ and $B$, the high-frequency gain of the transfer function $\hat{H}(s), b_0$, is equal to $b_p + \hat{A}_{2120}\theta$.

**Remark 1**

Since the uncertain linear system (1) is observable, we can always determine the state transformation matrix $\tilde{T}$ to be the one that transforms the pair $(\hat{A}, \hat{C})$ into the observable canonical form (see [38]). After $\tilde{T}$ is found out, we can design the disturbance transformation matrix $\hat{M}$ such that the matrices $D$ and $E$ are completely known. Here, $\hat{M}$ can be selected such that $\|\hat{M}\|_2$ is upper bounded by any desired positive constant for any admissible value of the unknown parameter vector $\theta$ (Assumption 3). This selection of $\hat{M}$ will guarantee a fixed disturbance attenuation level from $\hat{w}$ to the tracking error for the closed-loop adaptive system with the proposed reduced-order controller.

The design model (2) satisfies Assumptions 2–4.

**Assumption 2**

The matrix $E$ is such that $EE' > 0$.

Define $\zeta = (EE')^{-1/2}$, $L = DE'$, and $\tilde{L} = [0_{1 \times (r-1)}]$'. The assumption on the parameter $\theta$ is given below.

**Assumption 3**

The sign of the high-frequency gain $b_0$ is known. There exists a known smooth nonnegative radially unbounded strictly convex function $P: \mathbb{R}^\sigma \rightarrow \mathbb{R}$, such that $\theta$ belongs to the set $\Theta := \{\theta \in \mathbb{R}^\sigma : P(\theta) \leq 1\}$. Furthermore, $\forall \bar{\theta} \in \Theta$, we have $\text{sgn}(b_0)(b_p + \hat{A}_{2120}\bar{\theta}) > 0$.

The following assumption is made on the reference trajectory $y_d$ that $x_1$ is to track.
Assumption 4

The reference trajectory \( y_d \) is \( r \) times continuously differentiable. The signal \( y_d \) and the first \( r \) derivatives of \( y_d \) are available for feedback. Denote \( Y_d := [y_d \ y_d^{(1)} \ldots \ y_d^{(r)}] \) and \( Y_d(0) = [y_d(0) \ y_d^{(1)}(0) \ldots \ y_d^{(r-1)}(0)] \), where \( y_d^{(i)} \) is the \( i \)th derivative of \( y_d \), \( i = 1, \ldots, r \).

For system (1), the control law is generated by \( u(t) = \mu(y_{[0,t]}, Y_{d[0,t]}) \). Furthermore, it must satisfy the following condition. For any uncertainty \( (x_0, \theta, \dot{w}_{[0,\infty)}, Y_{d0}, Y_{d(r)}_{d[0,\infty)}) \in \mathcal{Y} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \), which consists of the initial condition of the system, the true value of the parameters, the waveform of the disturbance input, the initial condition of the reference trajectory, and the waveform of the \( r \)th order derivative of the reference trajectory, there must be a unique solution \( \tilde{x}_{[0,\infty)} \) for the closed-loop system, which results in a continuous control function \( u_{[0,\infty)} \). The class of these admissible controllers is denoted by \( \mathcal{M}_{u} \).

The control objective is to make the system output \( Cx \) to track the reference trajectory \( y_d \) asymptotically while attenuating the effect of the uncertainty \( (x_0, \theta, \dot{w}_{[0,\infty)}, Y_{d0}, Y_{d(r)}_{d[0,\infty)}), \) where the exogenous input \( \dot{w} \) can be taken to be any open-loop time function, as in the case of \( H^\infty \)-optimal control problems. A precise definition of the objective is further given below.

Definition 1

A controller \( \mu \in \mathcal{M}_{u} \) is said to achieve disturbance attenuation level \( \gamma \) if there exist a function \( l(t, \theta, x_{[0,t]}, Y_{d[0,t]}) \) and a constant \( l_0 \geq 0 \), such that, for all \( t_f \geq 0 \), the following inequality holds:

\[
\sup_{(x_0, \theta, \dot{w}_{[0,\infty)}, Y_{d0}, Y_{d(r)}_{d[0,\infty)}) \in \mathcal{Y}} J_{\gamma t} \leq 0
\]

and \( l(t, \theta, x(t), Y_{d[0,t]}), t_f \geq 0, \forall t \in [0, t_f] \), along the closed-loop trajectory, where

\[
J_{\gamma t} := \int_{0}^{t_f} \left( (x_1(\tau) - y_{d}(\tau))^2 + l(\tau, \theta, x(\tau), Y_{d[0,\tau]})) - \gamma^2 |w(\tau)|^2 \right) d\tau
\]

and \( -\gamma^2 [\theta' - \tilde{\theta}_0' \ x_0' - \tilde{x}_0' \] Q_{0}^{-1} l_{0}

where \( \tilde{\theta}_0 \in \Theta \) is the initial guess of the unknown parameter vector \( \theta \), \( \tilde{x}_0 \) is the initial guess of the unknown initial state \( x_0 \), the \( (n+\sigma) \times (n+\sigma) \)-dimensional matrix \( Q_0 > 0 \) is the weighting matrix on the initial estimation error, quantifying the level of confidence in the estimate \( [\tilde{\theta}_0' \ x_0'] \), and \( \tilde{Q}_0^{-1} \) admits the structure

\[
\begin{bmatrix}
Q_0^{-1} & Q_0^{-1} \Phi_0' \\
\Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1} \Phi_0'
\end{bmatrix}
\]

where \( Q_0 > 0 \) and \( \Pi_0 > 0 \) are \( \sigma \times \sigma \)- and \( n \times n \)-dimensional, respectively.

The problem formulated above is then brought into the framework of \( H^\infty \)-optimal control for affine-quadratic nonlinear systems with imperfect state measurements. The system dynamics (2) is expanded by adding the simple dynamics of \( \theta \): \( \dot{\theta} = 0 \). Let \( \zeta \) denote the expanded state vector.
\[ \dot{\xi} = [\theta' \; x']', \] which satisfies the following dynamics:

\[
\begin{bmatrix}
0_{2\times 2} & 0_{2\times 1} \\
A'_{211} + uA_{212} & 0_{2\times 2}
\end{bmatrix}
\begin{bmatrix}
\xi \\
u
\end{bmatrix} +
\begin{bmatrix}
0_{2\times 1} \\
B
\end{bmatrix} u +
\begin{bmatrix}
0_{2\times 1} & 0_{2\times 1}
\end{bmatrix} w =: A(y, u)\xi + Bu + \tilde{D} w
\]

(5a)

\[ y = [0_{1\times 2} \quad C] \xi + E w =: \tilde{C} \xi + E w \]

(5b)

The worst-case optimization of the cost function (3) can be carried out in two steps with the following inequality:

\[
\sup_{(x_0, 0, \bar{u}(0, \infty), Y_{d0}, \gamma_{d(0, \infty)}) \in \mathcal{W}} J_{y(t_f)} = \sup_{Y_{d0} \in \mathbb{R}^r, \gamma_{d(0, \infty)} \in \mathcal{C}, \gamma_{d(0, \infty)} \in \mathcal{C}} \sup_{(x_0, 0, \bar{u}(0, \infty), Y_{d0}, \gamma_{d(0, \infty)}) \in \mathcal{W}} J_{y(t_f)}
\]

(6)

The design procedure starts with the inner supremization, which can be interpreted as the evaluation of the worst-case performance with a known output waveform. This step is actually the estimator design step. Cost-to-come function analysis will be applied to derive a finite-dimensional estimator. The outer supremization can be interpreted as the computation of the worst-case measurement waveform against a given control law, which is crucial for the determination of achievability of objective (3). This step is the control design step carried out in Section 5.

4. ESTIMATOR DESIGN

In order to set up an appropriate basis for the discussion of controller design in the following section, we begin by reviewing the estimator design.

Given \( Y_{d0} \), the measurement waveform \( y(0, \infty) \), and the reference trajectory \( \gamma_{d(0, \infty)} \), the control waveform \( u(0, \infty) \) is also known. The cost function we consider at this step is

\[
J_{y(t_f)} = \int_0^{t_f} (|x_1(\tau) - \gamma_{d0}(\tau)|^2 + |\xi(\tau) - \tilde{\xi}(\tau)|^2) \tilde{Q}(\tau, y_{d(0, \infty)}) - \gamma^2 |w(\tau)|^2 + 2(\xi(\tau) - \tilde{\xi}(\tau)) \tilde{Q}(\tau, y_{d(0, \infty)}) d\tau
- \int_0^{t_f} \gamma^2 [l_1(\tau, 0, \tau, Y_{d(0, \infty)}, \gamma_{d(0, \infty)}, Y_{d(0, \infty)})] d\tau
+ 2 \gamma^2 [l_2(\tau, 0, \tau, Y_{d(0, \infty)}, \gamma_{d(0, \infty)}, Y_{d(0, \infty)})] d\tau
\]

(7)

where the second and the fourth terms are introduced as part of the function \( l \) in consideration of the robustness of the complete adaptive system. \( l_1 \) and \( l_2 \) are functions to be designed in this section; the term \( 2(\xi - \tilde{\xi})'l_2(d(0, \infty)) \) is added here to incorporate a soft-projection algorithm, which keeps \( \tilde{\theta} \) within a vicinity of \( \Theta \) such that \( \tilde{\theta}_0 \) is always positive; \( \tilde{\xi} \) is the worst-case estimate for the expanded state \( \xi \) and will be designed in the controller step; \( \tilde{Q} \) is the nonnegative-definite weighting function to be described shortly in this section. In (7), we do not include some terms of (4) which are constants at this step of optimization.

Expressing the cost function (7) in terms of the state vector \( \xi \), we are able to apply the cost-to-come function analysis (see [26,29] for details) for this affine quadratic problem to derive a finite
dimensional estimator. Then, we have the following value function:

\[ W(\xi, \bar{\xi}, \hat{\Sigma}) = |\bar{\xi} - \bar{\xi}|^2 \Sigma^{-1} \] (8)

where the covariance matrix \( \hat{\Sigma} \) and the estimate of \( \xi, \bar{\xi}, \) are given by

\[
\hat{\Sigma} = (\bar{\Sigma} - \bar{\xi}^2 \bar{L}\hat{C})\Sigma(\bar{\xi} - \bar{\xi}^2 \bar{L}\hat{C}) + \gamma^{-2} \bar{L}'\tilde{\bar{\Sigma}}^{-1}\Sigma(\gamma^2 \bar{C}'\bar{C} - \bar{C}'\hat{C} - \hat{Q})\Sigma
\]

\[
\hat{\Sigma}(0) = \gamma^{-2} \tilde{\bar{\Sigma}}^{-1}
\]

\[
\bar{\xi} = (\bar{\Sigma}(\bar{C}'\hat{C} + \hat{Q}))^{-1}\bar{\Sigma}(\hat{C}'y_d + \hat{Q}\bar{\xi}) + \hat{B}u + \gamma^2 (\bar{\Sigma}\hat{C}' + \hat{L})(y - \hat{C}\bar{\xi}), \quad \bar{\xi}(0) = \begin{bmatrix} \hat{\theta}_0 \\ \hat{x}_0 \end{bmatrix}
\]

To further deduce the structure of the estimator and the existence of \( \hat{\Sigma} \), we partition \( \bar{\xi} \) into \( [\hat{\theta}' \; \hat{x}']' \) and partition the covariance matrix \( \hat{\Sigma} \) as

\[
\hat{\Sigma} = \begin{bmatrix} \Sigma & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{bmatrix}
\]

Here, we introduce the quantities: \( \Phi := \Sigma_{21}\Sigma^{-1} \) and \( \Pi := \gamma^2(\Sigma_{22} - \Sigma_{21}\Sigma^{-1}\Sigma_{12}) \). Then we have \( \hat{\Sigma} > 0 \) if and only if \( \Sigma > 0 \) and \( \Pi > 0 \). \( \Sigma \) will play the role of the worst-case covariance matrix of the parameter estimation error.

To guarantee the boundedness of \( \Sigma \) and the robustness of the close-loop system, we select the weighting matrix \( \hat{Q} \) in (7) as

\[
\hat{Q}(t, y_{[0,t]}, Y_d_{[0,t]}) = (\hat{\Sigma}(t))^{-1}\begin{bmatrix} 0_{\sigma \times \sigma} & 0_{\sigma \times n} \\ 0_{n \times \sigma} & \Delta \end{bmatrix}(\hat{\Sigma}(t))^{-1} + \begin{bmatrix} \varepsilon(t)(\Phi(t))'C'(\gamma^2 \bar{\xi}^2 - 1)C\Phi(t) & 0_{\sigma \times n} \\ 0_{n \times \sigma} & 0_{n \times n} \end{bmatrix}
\]

\[
= \begin{bmatrix} -(\Phi(t))'I_n \Delta \Pi^{-1} \Delta \Pi^{-1} \left[ -(\Phi(t))'I_n \right] & \varepsilon(t)(\Phi(t))'C'(\gamma^2 \bar{\xi}^2 - 1)C\Phi(t) \\ 0_{n \times \sigma} & 0_{n \times n} \end{bmatrix}
\]

where \( \Delta = \gamma^{-2} \beta_\Delta \Pi + \Delta_1 \) with the constant \( \beta_\Delta \geq 0 \) and the \( n \times n \)-dimensional constant matrix \( \Delta_1 > 0 \); \( \varepsilon \) is a scalar function defined by

\[
\varepsilon(t) = \text{Tr}((\Sigma(t))^{-1})/K_c \quad \forall t \in [0, t_f]
\] (10)

In (10), \( K_c \) is a constant that is greater than or equal to \( \gamma^2 \text{Tr}(Q_0) \), corresponding to the preslected maximum level of the quantity \( \text{Tr}((\Sigma(t))^{-1}) \). Therefore, \( \varepsilon \leq 1 \); \( \Sigma \) is nonincreasing.

With the cost-to-come function analysis, \( \Sigma, \Phi, \) and \( \Pi \) have the following dynamics:

\[
\dot{\Sigma} = -(1 - \varepsilon)\Sigma\Phi'C'(\gamma^2 \bar{\xi}^2 - 1)C\Phi \Sigma, \quad \Sigma(0) = \gamma^{-2} Q_0^{-1}
\] (11a)

\[
\dot{\Phi} = (A - \bar{\xi}'LC - \Pi C'(\gamma^2 - \gamma^{-2})C)\Phi + y\hat{A}_{211} + u\hat{A}_{212}, \quad \Phi(0) = \Phi_0
\] (11b)

\[
\dot{\Pi} = (A - \bar{\xi}'LC + \beta_\Delta/2I_n)\Pi + \Pi(A - \bar{\xi}'LC + \beta_\Delta/2I_n)' - \Pi C'(\gamma^2 - \gamma^{-2})C\Pi + DD'
\]

\[
- \bar{\xi}'LL' + \gamma^2 \Delta_1, \quad \Pi(0) = \Pi_0
\] (11c)
We observe that for the matrix $\Sigma$ to be bounded, we need to select $\gamma$ such that $\gamma^2 \zeta^2 \geq 1$, namely, $\gamma \geq \zeta^{-1}$. This can be interpreted as that the achievable disturbance attenuation level is no smaller than the noise intensity in the measurement channel. The quantity $\zeta^{-1}$ is the ultimate lower bound on the achievable performance level for the adaptive system using the proposed method in this paper. We make the following assumption to guarantee this property.

**Assumption 5**

If the matrix $A - \zeta^2 LC$ is Hurwitz, then choose the desired disturbance attenuation level $\gamma > \zeta^{-1}$. In case $\gamma = \zeta^{-1}$, choose $\beta_{\Delta} > 0$ such that $A - \zeta^2 LC + \beta_{\Delta} / 2 I_n$ is Hurwitz. If the matrix $A - \zeta^2 LC$ is not Hurwitz, then choose $\gamma > \zeta^{-1}$.

To simplify the estimator structure, we initialize the $\Pi$ dynamics as its steady-state value. Here, we choose the initial weighting matrix $P_0$ in (11c) to be the unique positive-definite solution to the algebraic Riccati equation:

$$
(A - \zeta^2 LC + \beta_{\Delta} / 2 I_n)\Pi + \Pi + (A - \zeta^2 LC + \beta_{\Delta} / 2 I_n)' - \Pi C' (\zeta^2 - \gamma^{-2}) C \Pi + D D' = 0
$$

Then, $\Pi$ will be a positive-definite constant matrix. Furthermore, we have that the matrix $A_f := A - \zeta^2 LC - \Pi C' (\zeta^2 - \gamma^{-2}) C$ is Hurwitz (see [39]).

**Remark 2**

Consider the dynamic equation (11a) for the covariance matrix $\Sigma$. With $\gamma \geq \zeta^{-1}$, $Q_0 > 0$, (10) holding, and $K_c \geq \gamma^2 \text{Tr}(Q_0)$. The matrix $\Sigma$ is upper and lower bounded as follows: whenever it exists on $[0, t_f]$ and $\Phi$ is continuous on $[0, t_f]$.

$$
K_c^{-1} I_\sigma \leq \Sigma(t) \leq \gamma^{-2} Q_0^{-1}, \quad \gamma^2 \text{Tr}(Q_0) \leq \text{Tr}((\Sigma(t))^{-1}) \leq K_c \quad \forall t \in [0, t_f]
$$

On the basis of Assumption 3, we aim at keeping $\tilde{\theta}$ within a vicinity of $\Theta$ such that $\tilde{\theta}_0$ is always positive. Here, we introduce a smooth-projection algorithm, which was first proposed in [32] and also adopted in many other works [34–37, 40]. Define

$$
\rho := \inf\{P(\tilde{\theta}) | \tilde{\theta} \in R^\sigma \text{ and } b_{\rho_0} + \tilde{A}_{2120} \tilde{\theta} = 0\}
$$

then, $\rho > 1$. Fix any $\rho_0 \in (1, \rho)$. The vicinity of $\Theta$ is then constructed as an open set $\Theta_{\rho_0} := \{\tilde{\theta} \in R^\sigma | P(\tilde{\theta}) < \rho_0\}$. The design intends to guarantee that the estimate $\tilde{\theta}$ always lies in $\Theta_{\rho_0}$. Then, we have that the estimate $\tilde{\theta}_0 := b_{\rho_0} + \tilde{A}_{2120} \tilde{\theta}$ is bounded away from 0. The smooth-projection function is then defined by

$$
P_r(\tilde{\theta}) := \begin{cases}
\exp((1 - P(\tilde{\theta}))^{-1}(\rho_0 - P(\tilde{\theta}))^{-3} (\frac{\partial P}{\partial \tilde{\theta}}(\tilde{\theta})))' & \forall \tilde{\theta} \in \Theta_{\rho_0} \setminus \Theta \\
0_{\sigma \times 1} & \forall \tilde{\theta} \in \Theta
\end{cases}
$$

It is obvious that $P_r(\tilde{\theta})$ and $p_r(\tilde{\theta})$ are smooth on the set $\Theta_{\rho_0}$, and $(\theta - \tilde{\theta})' P_r(\tilde{\theta}) \leq 0, \forall \tilde{\theta} \in \Theta_{\rho_0}$.
This soft-projection algorithm is incorporated into the cost function (7) by setting $l_1$ to be $\tilde{\zeta}$ and $l_2$ to be $[-(P_r(\tilde{\theta}))^{\prime} 0_{1 \times n}]$. Then, the dynamics of $\tilde{\zeta}$ has the formula below after incorporating a projection term:

$$
\dot{\tilde{\zeta}} = -\tilde{\Sigma}[(P_r(\tilde{\theta}))^{\prime} 0_{1 \times n}^{\prime}] + \hat{A}\tilde{\zeta} - \tilde{\Sigma}\hat{C}^{\prime}(y_d - \hat{C}\tilde{\zeta}) + \hat{B}u - \tilde{\Sigma}\hat{Q}\tilde{\zeta}_c \\
+ (\gamma^2\tilde{\Sigma}\hat{C}^{\prime} + \tilde{L})\tilde{z}^2(y - \hat{C}\tilde{z}), \quad \tilde{\zeta}(0) = [\tilde{\theta}_0^{\prime} \tilde{x}_0^{\prime}]^{\prime}
$$

(15)

The projection term in (15) leads to a term $2(\theta - \tilde{\theta})P_r(\tilde{\theta})$ in the derivative of the value function $W$. The term $2(\theta - \tilde{\theta})P_r(\tilde{\theta})$ is nonpositive, $\forall \tilde{\theta} \in \Theta_o$, zero on the set $\Theta$, and tends to $-\infty$ as $\tilde{\theta}$ approaches the boundary of the set $\Theta_o$. The time derivative of the value function $W$ along the solution of $\tilde{\zeta}$, $\tilde{\zeta}$, and $\tilde{\Sigma}$ is given by

$$
\dot{W}(\tilde{\zeta}, \tilde{\xi}, \tilde{\Sigma}, y_d, u, w) = -|x_1 - y_d|^2 - \gamma^4|x - \hat{x} - \Phi(\theta - \tilde{\theta})|_{\Pi - 1}\Delta \Pi + 2(\theta - \tilde{\theta})P_r(\tilde{\theta})
$$

$$
- \epsilon(\gamma^2\tilde{z}^2 - 1)|\theta - \tilde{\theta}|^{\prime}_{\Phi C^{\prime} C} + |C\tilde{x} - y_d|^2
$$

$$
+ |\zeta_c|^2(\Phi, s\Sigma) - \gamma^2\tilde{z}^2|y - C\hat{x}|^2 + \gamma^2|w|^2
$$

$$
- \gamma^2|w - w_*(\zeta, \tilde{\zeta}, \tilde{\Sigma}, w)|^2
$$

(16)

where $w_*$ denotes the worst-case disturbance for the control design and is given by

$$
w_*(\zeta, \tilde{\zeta}, \tilde{\Sigma}, w) = \tilde{z}^2E'(y - \hat{C}\tilde{z}) + \gamma^{-2}(I_{q0} - \tilde{z}^2E'E)\tilde{D}'\tilde{S}^{-1}(\zeta - \tilde{\zeta})
$$

(17)

(16) holds as long as $\Sigma > 0$ and $\tilde{\theta} \in \Theta_o$.

We make a summary of the estimator dynamics below for ease of reference:

$$
\dot{\hat{\Sigma}} = -(1 - \epsilon)\Sigma\Phi^{\prime}C^{\prime}(\gamma^2\tilde{z}^2 - 1)C\Phi\Sigma, \quad \Sigma(0) = \gamma^{-2}Q_0^{-1}
$$

(18a)

$$
\dot{s\Sigma} = (\gamma^2\tilde{z}^2 - 1)(1 - \epsilon)C\Phi^{\prime}C, \quad s\Sigma(0) = \gamma^2\text{Tr}(Q_0)
$$

(18b)

$$
\dot{\Phi} = A_f\Phi + y\tilde{A}_{211} + u\tilde{A}_{212}, \quad \Phi(0) = \Phi_0
$$

(18c)

$$
\dot{\tilde{\theta}} = -\Sigma P_r(\tilde{\theta}) - \Phi\Sigma\Phi^{\prime}C^{\prime}(y_d - C\tilde{x}) - [\Sigma \Phi\Phi^{\prime}]\tilde{Q}\zeta_c + \gamma^2\tilde{z}^2\Sigma \Phi\Phi^{\prime}C^{\prime}(y - C\hat{x}), \quad \tilde{\theta}(0) = \tilde{\theta}_0
$$

(18d)

$$
\dot{x} = -\Phi\Sigma P_r(\tilde{\theta}) + Ax - (\gamma^2\Pi + \Phi\Sigma\Phi^{\prime})C^{\prime}(y_d - C\tilde{x}) - [\Phi\Sigma \gamma^{-2}\Pi + \Phi\Sigma\Phi^{\prime}]\tilde{Q}\zeta_c
$$

$$
+ Bu + (y\tilde{A}_{211} + u\tilde{A}_{212})\tilde{\theta} + \tilde{z}^2(\Pi C^{\prime} + \gamma^2\Phi\Sigma\Phi^{\prime}C^{\prime} + L)(y - C\hat{x}), \quad \dot{x}(0) = \dot{x}_0
$$

(18e)
where $s_\Sigma$ is a scalar signal defined as $s_\Sigma(t):=\text{Tr}((\Sigma(t))^{-1})$, which is introduced to avoid the computation of $(\Sigma(t))^{-1}$ on line; $\xi_c=\hat{\xi}-\hat{\xi}$ will be designed at the end of control design.

Assuming that all signals exist, $\Sigma(t)>0$ and $\hat{\theta}(t)\in\Theta_a$, $\forall t\in[0,t_f]$, the cost function can equivalently be expressed as

$$J_{\gamma t_f} = -|\theta - \hat{\theta}(t_f)|^2 (\Sigma(t_f))^{-1} - \gamma^2|x(t_f) - \tilde{x}(t_f) - \Phi(t_f)(\theta - \hat{\theta}(t_f))|^2_{\Pi^{-1}}$$

$$+ \int_0^{t_f} (|C\tilde{x}(\tau) - y_d(\tau)|^2 + |\xi_c(\tau)|^2_{\tilde{Q}(\Phi(\tau), s_\Sigma(\tau))} - \gamma^2\xi^2 |y(\tau) - C\tilde{x}(\tau)|^2 - \gamma^2|w(\tau) - w_\eta(\hat{\xi}(\tau), \hat{\xi}(\tau), \hat{\Sigma}(\tau), w(\tau))|^2) \, d\tau$$

(19)

In actual implementation of the estimator, if the system contains many unknown parameters, we may alternatively generate $\Phi$ using the 2n-dimensional pre-filtering system below for $y$ and $u$, instead of using an $n$ integrators on line (similar to the reasoning in [28]). This way, the estimator dynamics may be simplified by $n(\sigma-2)$ integrators:

$$\dot{\eta} = A_f \eta + p_n y, \quad \eta(0) = 0_{n \times 1} \quad (20a)$$

$$\dot{\lambda} = A_f \lambda + p_n u, \quad \lambda(0) = 0_{n \times 1} \quad (20b)$$

where $p_n$ is an $n$-dimensional vector such that $(A_f, p_n)$ is controllable. Then, $\Phi$ is given by an algebraic relation in terms of $\eta$ and $\lambda$:

$$\Phi = [A_f^{n-1} \eta \quad \ldots \quad A_f \eta \quad \eta] M_f^{-1} \tilde{A}_{211} + [A_f^{n-1} \lambda \quad \ldots \quad A_f \lambda \quad \lambda] M_f^{-1} \tilde{A}_{212}$$

(21)

Remark 3

The derived estimator via cost-to-come function analysis involves the dynamics listed in (18a)–(18e). According to (10), we can simply define $\varepsilon\equiv 1$ by setting $K_c=\gamma^2\text{Tr}(Q_0)$. Then we observe, from (18a) and (18b), $\Sigma$ and $s_\Sigma$ remain constant on $[0, \infty)$, which simplifies the estimator structure by $\sigma(\sigma+1)/2+1$ integrators. Plus the simplification about $\Phi$ as we mentioned above, the estimator will only include the dynamics of $\hat{\theta}$, $\hat{x}$, $\eta$, and $\lambda$ (see (18d), (18e), (20a) and (20b)). In this case, the estimator order is $3n + \sigma$.

On the basis of the result of this step, we now turn to the controller design step in the next section.

5. REDUCED-ORDER ADAPTIVE CONTROLLER DESIGN

We describe in this section the reduced-order controller design for the uncertain system. In view of the equivalent form (19) of the cost function obtained in the previous section, we can rewrite
Inequality (6) to be
\[
\sup_{(x_0, \hat{\theta}, \hat{Y}(0), Y_d(0), Y_d(0), Y_d(0)) \in \mathcal{W}'} J_{yt}\left(\int_0^{t_f} (|C\dot{x}(t)| - y_d(t)|^2 + |\xi_c(t)|^2) d\tau + \tilde{t}(t, y(0), Y_d(0), Y_d(0)) - \gamma^2 \xi_2^2|y(t) - C\dot{x}(t)|^2 d\tau - l_0 \right)
\]
where \(\tilde{t} = t - \xi - \tilde{\xi}_Q + 2(\tilde{\theta} - \bar{\theta})^T P_r(\bar{\theta})\) is to be designed in this section; and the last inequality holds if all closed-loop signals exist, \(\Sigma(t) > 0\) and \(\bar{\theta}(t) \in \Theta_\alpha\), on \([0, t_f]\). We notice that the dynamics of the signals \(\dot{x}, \bar{\theta}, \Phi, \Sigma, y_d, \) and \(s_\Sigma\) in (22) are driven by measurement \(y\), reference input \(Y_d\), control input \(u\), or the worst-case estimate \(\tilde{\xi}\), where \(y\) and \(Y_d\) are measurable, \(u\) and \(\tilde{\xi}\) to be constructed in this section. Therefore, the control design problem turns out to be a \(H^\infty\)-optimal control problem with full-information measurement. The design aims to guarantee that the supremum in (22) is less than or equal to zero for all measurement waveforms.

A new variable \(v\) is introduced, instead of considering \(y\) as the maximizing variable, to reveal the suitable dynamic structure for backstepping design. The variable \(v\) is formulated as \(v := \xi(y - C\dot{x})\). In terms of \(v\), we express the relevant dynamics ((18a)--(18e)) obtained in the estimator design in the form of various functions on the independent variables ((23a)--(23f)) rather than their actual definitions. We will denote \(\tilde{x} = [\tilde{x}_1 \ldots \tilde{x}_n]'\) and \(\Phi = [\Phi_1' \ldots \Phi_n']\), where \(\tilde{x}_i, i = 1, \ldots, n\), are scalars and \(\Phi_i, i = 1, \ldots, n\), are row vectors:

\[
\dot{s}_c = (\gamma^2 \xi_2^2 - 1)(1 - \hat{e}) \Phi_1 \Phi_1'
\]

(23a)

\[
\dot{\Sigma} = -(1 - \hat{e})\Sigma \Phi_1' (\gamma^2 \xi_2^2 - 1) \Phi_1 \Sigma
\]

(23b)

\[
\dot{\theta} = \delta(y_d, \tilde{x}_1, \Phi_1, \tilde{\theta}, \Sigma) + \varphi(\bar{\Phi}, \bar{\Sigma}) \bar{Q} \xi_c + \kappa(\Phi_1, \Sigma) v
\]

(23c)

\[
\dot{x}_i = f_i(y_d, \tilde{x}_1, \ldots, \tilde{x}_i, \bar{\theta}, \Phi_1, \Phi_i, \Sigma) + x_{i+1} + g_i(\bar{\Phi}, \Sigma) \bar{Q} \xi_c
\]

\[
+ h_i(\tilde{\theta}, \Phi_1, \Phi_i, \Sigma) v, \quad i = 1, \ldots, r - 1
\]

(23d)

\[
\dot{x}_r = f_r(y_d, \tilde{x}_1, \ldots, \tilde{x}_r, \bar{\theta}, \Phi_1, \Phi_r, \Sigma) + a_{r+1} x_{r+1} + \cdots + a_r x_r + (b_{p0} + \bar{A}_{2120} \bar{\theta}) u
\]

\[
+ g_r(\bar{\Phi}, \Sigma) \bar{Q} \xi_c + h_r(\tilde{\theta}, \Phi_1, \Phi_r, \Sigma) v
\]

(23e)

\[
\dot{\Phi}_i = \psi_i(\tilde{x}_1, \Phi_1, \ldots, \Phi_i) + \Phi_{i+1} + g_{i,n}' x_n + \bar{A}_{2111} \xi_c\]

(23f)

where the nonlinear functions \(f_i, i = 1, \ldots, r\), and \(\delta\) are smooth as long as \(\tilde{\theta} \in \Theta_\alpha\); the nonlinear functions \(\varphi, \kappa, h_i, g_i, i = 1, \ldots, r\), are smooth; and the functions \(\psi_i, i = 1, \ldots, r\), are linear.
We will employ the integrator backstepping methodology [10] to design the control law \( u \). In view of a nonnegative definite weighting of \( \xi_c \) in the cost function (22), the design for \( \xi_c \) will be carried out last. Therefore, we, for convenience, set \( \xi_c \) to be zero first in this backstepping procedure, which does not influence the result of backstepping design.

During the backstepping design procedure, we will apply the following backstepping lemma to each individual step, which will relieve us from the tedious derivation work at each step.

**Lemma 1**

Let \( n_1, n_3, q \in \mathbb{N} \), \( k \in \mathbb{N} \cup \{0\} \), \( D_o \subseteq \mathbb{R}^{n_1} \) be nonempty and open, \( D_a \subseteq \mathbb{R}^{n_3} \) be nonempty and open, \( D_d \subseteq \mathbb{R}^{n_1} \) be nonempty and open, \( D_u \subseteq \mathbb{R}^{q} \) that contains a nonempty open subset of \( \mathbb{R}^{q} \), and \( D_1 \subseteq D_o \times D_d \) be nonempty. Let \( f_o, h_o, f_a, g_a \), and \( h_a \) be mappings of \( D_o \times D_a \times D_d \) into \( \mathbb{R}^{n_1} \), \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_3} \), \( \mathbb{R}^{n_3} \), \( \mathbb{R}^{n_1} \), and \( \mathbb{R}^{1} \times \mathbb{R}^{q} \), respectively. Let \( f_o \) and \( h_o \) be \( \mathcal{C}_k \) and all of their partial derivatives of \( k \)th order are further continuously differentiable with respect to \( x_o \in D_o \). Let \( f_a, g_a \), and \( h_a \) be \( \mathcal{C}_k \), \( g_a(x_o, x_a, x_d) \neq 0 \), \( \forall(x_o, x_a, x_d) \in D_o \times D_a \times D_d \), \( l_o: D_1 \rightarrow \mathbb{R} \) be continuous, \( \gamma > 0 \) and \( \gamma \in \mathbb{R} \), \( V_o: D_o \rightarrow \mathbb{R} \) be \( \mathcal{C}_{k+1} \), \( x_o: D_o \rightarrow D_d \) be \( \mathcal{C}_{k+1} \), \( \sigma_o: D_o \times D_d \rightarrow \mathbb{R}^{q} \) be \( \mathcal{C}_k \) and defined by

\[
\sigma_o(x_o, x_d) := \frac{1}{2\gamma^2} \left( h_o(x_o, x_o(x_d), x_d) \right) \left( \frac{\partial V_o}{\partial x_o}(x_o) \right)
\]

\( \phi: D_o \times D_a \times D_d \rightarrow \mathbb{R} \) be \( \mathcal{C}_k \), and \( \delta: D_a \times D_d \rightarrow \mathbb{R} \) be \( \mathcal{C}_{k+1} \). \( \phi \) and \( \delta \) are design functions. Assume that \( \delta \) satisfies the following two conditions.

(i) \( \delta(x_o, x_d) > 0, \forall(x_o, x_d) \in D_o \times D_d \).

(ii) \( p\delta(x_o, x_d) := \delta(x_o, x_d) + (\frac{\partial \delta}{\partial x_o}(x_o, x_d) (x_d - x_o(x_d))) > 0, \forall(x_o, x_d) \in D_o \times D_d \).

Let \( V: D_o \times D_d \rightarrow \mathbb{R} \) be defined by \( V(x_o, x_d) := V_o(x_o) + (\delta(x_o, x_d)(x_d - x_o(x_d)))^2 \), which is \( \mathcal{C}_{k+1} \).

Consider the system dynamics:

\[
\dot{x}_o(t) = f_o(x_o(t), x_a(t), x_d(t)) + h_o(x_o(t), x_a(t), x_d(t))w(t) \tag{25a}
\]

\[
\dot{x}_a(t) = f_a(x_o(t), x_a(t), x_d(t)) + g_a(x_o(t), x_a(t), x_d(t))u(t) + h_a(x_o(t), x_a(t), x_d(t))w(t) \tag{25b}
\]

where \( w(\cdot) \) is a continuous signal taking values in \( D_w \), \( x_d(\cdot) \) is a continuous signal taking values in \( D_d \), and \( u(\cdot) \) is a continuous signal taking values in \( \mathbb{R} \).

Assume that the derivative of \( V_o(x_o(t)) \) along a solution of the dynamics (25a) with \( x_d(t) = x_o(x_o(t)) \) can be expressed as, \( \forall(x_o, x_d) \in D_1, \forall w \in D_w \),

\[
\dot{V}_o(x_o, x_a, x_d, w)|_{x_o=x_o(x_d)} = -l_o(x_o, x_d) + \gamma^2|w|^2 - \gamma^2|w - \sigma_o(x_o, x_d)|^2 \tag{26}
\]

Then, there exists a \( \mathcal{C}_k \) function \( z: D_o \times D_a \times D_d \rightarrow \mathbb{R} \) given by (A2) such that the derivative of \( V(x_o(t), x_a(t)) \) along a solution of the dynamics (25) with \( u(t) = z(x_o(t), x_a(t), x_d(t)) \) can be expressed as, \( \forall(x_o, x_d) \in D_1, \forall x_a \in D_a, \forall w \in D_w \),

\[
\dot{V}(x_o, x_a, x_d, u, w)|_{u=z(x_o, x_a, x_d)} = -l_o(x_o, x_d) - \phi(x_o, x_a, x_d)(x_d - x_o(x_o)) + \gamma^2|w|^2 - \gamma^2|w - \sigma(x_o, x_a, x_d)|^2 \tag{27}
\]
where \( \sigma: \mathbb{D}_o \times \mathbb{D}_a \times \mathbb{D}_d \rightarrow \mathbb{R}^d \) is \( \mathcal{E}_k \) and given by, with \( x := [x'_o \, x'_a]' \),
\[
\sigma(x_o, x_a, x_d) := \frac{1}{2\gamma^2} \begin{bmatrix} h_o(x_o, x_a, x_d) \\ h_a(x_o, x_a, x_d) \end{bmatrix} \left( \frac{\partial V}{\partial x}(x_o, x_a) \right)'
\]
(28)

If, in addition, there exists \( (x_o, x_a, x_d) \in \mathbb{D}_o \times \mathbb{D}_a \times \mathbb{D}_d \), such that \( (\nabla V_o/\nabla x_o)(x_o) = 0_{n_1 \times 1} \), \( f_o(x_o, x_a, x_d) = 0_{n_1 \times 1} \), \( f_a(x_o, x_a, x_d) = 0 \), \( x_o(x_o) = x_o \), and \( \phi(x_o, x_a, x_d) = 0 \), then \( \sigma(x_o, x_a, x_d) = 0 \).

**Proof**
See Appendix A.1 for details.

The design is proceeded under two cases: \( r = 1 \) and \( r > 1 \). First, we consider the case \( r = 1 \).

**Step 1** At this step, we define the transformed variable \( z_1 = x_1 - y_d \). For notational consistency, we choose \( z_0 = y_d \) and \( V_0 = 0 \). Then in the application of Lemma 1, we make the following substitution:
\[
X_{1o} := [y_d \ \dot{\theta} \ \Sigma'] \rightarrow x_o, \quad X_{1d} := \dot{x}_1 \rightarrow x_a, \quad v \rightarrow w, \quad u \rightarrow u
\]
\[
X_{1d} := [y^{(1)}_d \ \Phi_1 \ \dot{x}_2 \ \ldots \ \dot{x}_n \ \Phi_2 \ \ldots \ \Phi_n]' \rightarrow x_d, \quad \mathbb{D}_{1a} := \mathbb{R} \rightarrow \mathbb{D}_a
\]
\[
\mathbb{D}_{1o} := \{X_{1o} : \dot{\theta} \in \Theta_o\} \rightarrow \mathbb{D}_o, \quad \mathbb{D}_{1d} := \mathbb{R}^{1+n\sigma+n-1} \rightarrow \mathbb{D}_d, \quad \mathbb{D}_w := \mathbb{R} \rightarrow \mathbb{D}_w, \quad \gamma \rightarrow \gamma
\]
\[
\mathbb{D}_{1o} \times \mathbb{D}_{1d} \rightarrow \mathbb{D}_1, \quad \text{Any number in } \mathbb{N} \rightarrow k, \quad V_0 \rightarrow V_o, \quad 0 \rightarrow \sigma_o, \quad x_0 \rightarrow x_o, \quad 0 \rightarrow l_o
\]
\[
F_{1o} := \begin{bmatrix} y^{(1)}_d \\ \dot{\theta} \\ \Sigma' \end{bmatrix} \rightarrow f_o, \quad H_{1o} := [0 \ \dot{\theta}_{\sigma \times \sigma} \ 0]' \rightarrow h_o
\]
\[
F_{1a} := f_1 + \tilde{x}_2 \rightarrow f_a, \quad G_{1a} := b_{p_0} + \tilde{A}_{2126} \dot{\theta} \rightarrow g_a, \quad H_{1a} := h_1 \rightarrow h_a
\]
Choose a constant \( \gamma_1 > 0 \) and a smooth design function \( \beta_1 : \mathbb{D}_{1o} \times \mathbb{D}_{1a} \times \mathbb{D}_{1d} \rightarrow \mathbb{R} \) such that
\[
\beta_1(X_{1o}, X_{1a}, X_{1d}) \geq c_{\beta_1} > 0 \quad \forall (X_{1o}, X_{1a}, X_{1d}) \in \mathbb{D}_{1o} \times \mathbb{D}_{1a} \times \mathbb{D}_{1d}
\]
(29)
where \( c_{\beta_1} \) is a constant. Again, we make the following substitution by Lemma 1:
\[
\sqrt{\gamma_1} \rightarrow \delta, \quad (1 + \beta_1)z_1 \rightarrow \phi, \quad V_1 \rightarrow V, \quad \mu \rightarrow \lambda
\]
(30)
Then, \( V_1 = \gamma_1^2 z_1 \), \( V_1 : \mathbb{D}_{1o} \times \mathbb{D}_{1a} \rightarrow \mathbb{R} \), and \( \mu : \mathbb{D}_{1o} \times \mathbb{D}_{1a} \times \mathbb{D}_{1d} \rightarrow \mathbb{R} \) are smooth such that
\[
\dot{V}_1(X_{1o}, X_{1a}, X_{1d}, u, v) = -\gamma_1^2 - \beta_1 z_1^2 + \gamma_1^2 v^2 - (v - v_1(X_{1o}, X_{1a}, X_{1d}))^2
\]
\[
\forall X_{1o} \in \mathbb{D}_{1o} \quad \forall X_{1a} \in \mathbb{D}_{1a} \quad \forall X_{1d} \in \mathbb{D}_{1d} \quad \forall v \in \mathbb{D}_w
\]
where \( v_1 : \mathbb{D}_{1o} \times \mathbb{D}_{1a} \times \mathbb{D}_{1d} \rightarrow \mathbb{R} \) is smooth and can be easily obtained by Lemma 1.

The control function can be derived by (A2) as follows:

\[ u = \mu(X_{10}, X_{1a}, X_{1d}) = \frac{1}{b_0} \left( -f_1 - \ddot{x}_2 + y_d^{(1)} - \frac{1}{2\gamma^2} \gamma_1 h_1 h'_1 z_1 - \frac{1}{2} (1 + \beta) z_1 \right) \]  

(31)

We denote the value function \( V = V_1 \).

This completes the backstepping design for the case \( r = 1 \).

Next, we consider the backstepping design for the case \( r > 1 \). By repeated application of Lemma 1, we may derive the control law.

**Step 1**: Define the transformed variable \( z_1 = \ddot{x}_1 - y_d \). Similar as the case \( r = 1 \), we first choose \( x_0 = y_d \) and \( V_0 = 0 \). By Lemma 1, we make the following substitution:

\[
X_{10} := [y_d \ \ddot{\Theta} \ \sum s_{\Sigma}]' \rightarrow x_o, \quad X_{1a} := \ddot{x}_1 \rightarrow x_a, \quad X_{1d} := [y_d^{(1)} \ \Phi_1]' \rightarrow x_d
\]

\[ v \rightarrow w, \quad \ddot{x}_2 \rightarrow u, \quad \mathcal{D}_{10} := \{X_{10} : \ddot{\Theta} \in \Theta_o\} \rightarrow \mathcal{D}_o, \quad \mathcal{D}_{1a} := \mathbb{R} \rightarrow \mathbb{R}
\]

\[ \mathcal{D}_{1d} := \mathbb{R}^{1+\sigma} \rightarrow \mathcal{D}_d, \quad \mathcal{D}_w := \mathbb{R} \rightarrow \mathcal{D}_w, \quad \mathcal{D}_{1a} \times \mathcal{D}_{1d} \rightarrow \mathcal{D}_1, \quad \gamma \rightarrow \gamma
\]

Any number in \( \mathbb{N} \rightarrow k, \quad V_0 \rightarrow V_o, \quad 0 \rightarrow \sigma_o, \quad x_0 \rightarrow x_o, \quad 0 \rightarrow l_o
\]

\[ F_{1o} := \left[ \begin{array}{c} y_d^{(1)} \\ \delta \\ \frac{-(1 - \varepsilon) \sum_{\Phi_1} (\gamma^2 z^2 - 1) \Phi_1 \Sigma}{(\gamma^2 z^2 - 1)(1 - \varepsilon) \Phi_1 \Sigma} \end{array} \right] \rightarrow f_0, \quad H_{1o} := [0 \ \kappa' \ \ddot{\Theta}_{\sigma \times \sigma}]' \rightarrow h_o
\]

\[ F_{1a} := f_1 \rightarrow f_a, \quad G_{1a} := 1 \rightarrow g_a, \quad H_{1a} := h_1 \rightarrow h_a
\]

Here, we choose a constant \( \gamma_1 > 0 \) and a smooth design function \( \beta_1 : \mathcal{D}_{10} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \rightarrow \mathbb{R} \) in a way same as the case \( r = 1 \) (29). Again, we make the substitution by Lemma 1:

\[ \sqrt{\gamma_1} \rightarrow \delta, \quad (1 + \beta_1) z_1 \rightarrow \phi, \quad V_1 \rightarrow V, \quad x_1 \rightarrow x
\]

Then, \( V_1 = \gamma_1 z_1^2, \quad V_1 : \mathcal{D}_{10} \times \mathcal{D}_{1a} \rightarrow \mathbb{R} \), and \( \chi_1 : \mathcal{D}_{10} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \rightarrow \mathbb{R} \) are smooth and

\[ \dot{V}_1(X_{10}, X_{1a}, X_{1d}, \ddot{x}_2, v) |_{\ddot{x}_2 = x_1(X_{10}, X_{1a}, X_{1d})} = -z_1^2 - \beta_1 z_1^2 + \gamma^2 v^2 - \gamma^2 (v - v_1(X_{10}, X_{1a}, X_{1d}))^2
\]

\[ \forall X_{10} \in \mathcal{D}_{10}, \quad \forall X_{1a} \in \mathcal{D}_{1a}, \quad \forall X_{1d} \in \mathcal{D}_{1d}, \quad \forall v \in \mathcal{D}_w
\]

where \( v_1 : \mathcal{D}_{10} \times \mathcal{D}_{1a} \times \mathcal{D}_{1d} \rightarrow \mathbb{R} \) is smooth and appropriately defined in Lemma 1.

If \( \ddot{x}_2 \) had been the actual control input, then we would have used the following virtual control law \( \ddot{x}_2 = x_1(X_{10}, X_{1a}, X_{1d}) \) to guarantee the dissipation inequality with supply rate \( -z_1^2 - \beta_1 z_1^2 + \gamma^2 v^2 \).

This completes the first step of the backstepping design.

**Step i (1 < i < r)**: We use Lemma 1 to finish the design at this step. Assume that we have completed \( i - 1 \) steps of the backstepping procedure, and obtained, for \( j = 1, \ldots, i - 1 \),

\[ X_{ja} := [y_d \ \ddot{\Theta} \ \sum s_{\Sigma} \ \ddot{x}_1 \ y_d^{(1)} \ \Phi_1 \ \ldots \ \ddot{x}_{j-1} \ y_d^{(j-1)} \ \Phi_{j-1}] \]

(32a)

\[ \mathcal{D}_{ja} := \{X_{ja} : \ddot{\Theta} \in \Theta_o\} \]

(32b)
\( X_{ja} := \tilde{x}_j, \quad \mathcal{D}_{ja} := \mathbb{R} \)  
\( X_{jd} := [y_{d}^{(j)} \Phi_j], \quad \mathcal{D}_{jd} := \mathbb{R}^{1+\sigma} \)  
\( x_j : \mathcal{D}_{ja} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R} \)  
\( \beta_j : \mathcal{D}_{ja} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}, \quad \beta_j(X_{ja}, X_{ja}, X_{jd}) \geq c \beta_j > 0, \quad c \beta_j \in \mathbb{R} \)  
\( v_{i-1} : \mathcal{D}_{i-1o} \times \mathcal{D}_{i-1a} \times \mathcal{D}_{i-1d} \to \mathbb{R} \)  
where the nonlinear functions \( x_j, \beta_j, j = 1, \ldots, i - 1 \), are smooth on their domain of definition; \( F_{i-1o}, H_{i-1o}, F_{i-1a}, G_{i-1a}, H_{i-1a} \) and \( v_{i-1} \) are smooth mappings of \( \mathcal{D}_{i-1o} \times \mathcal{D}_{i-1a} \times \mathcal{D}_{i-1d} \); \( V_{i-1} \) is a smooth function on its domain of definition.

At step \( i \), define \( z_i = \tilde{x}_i - x_{i-1}(X_{i-1o}, X_{i-1a}, X_{i-1d}) \). Again, we apply Lemma 1 to make the following substitutions:

\[ X_{io} := [X_{i-1o}' X_{i-1a}' X_{i-1d}'] \to x_o, \quad X_{ia} := \tilde{x}_i \to x_a, \quad X_{id} = [y_{d}^{(i)} \Phi_i]' \to x_d \]
\[ \mathcal{D}_{io} := \mathcal{D}_{i-1o} \times \mathcal{D}_{i-1a} \times \mathcal{D}_{i-1d} = \{ X_{io} : \tilde{\theta} \in \Theta_o \} \to D_o, \quad \mathcal{D}_{ia} := \mathbb{R} \to D_a \]
\[ v \to w, \quad \tilde{x}_{i+1} \to u, \quad \mathcal{D}_{id} := \mathbb{R}^{1+\sigma} \to D_{id} \to D_{i} \to D_{i}, \quad V_{i-1} \to V_{o} \]
\[ v_{i-1} \to \sigma_o, \quad x_{i-1} \to x_o, \quad \tilde{z}_i^2 + \sum_{j=1}^{i-1} \beta_j z_j^2 \to l_o, \quad \text{Any number in } \mathbb{N} \to k, \quad \gamma \to \gamma \]

\[ F_{io} := [F_{i-1o}' F_{i-1a} G_{i-1a} \tilde{x}_i y_{d}^{(i)} \psi_{i-1} + \Phi_i]' \to f_o, \quad F_{ia} := f_i \to f_a \]
\[ H_{io} := [H_{i-1o}' H_{i-1a} 0 e_{n,i} \tilde{A}_{211a} \psi_{i-1}^{-1}]' \to h_o, \quad G_{ia} := 1 \to g_a, \quad H_{ia} := h_i \to h_a \]
Note that $X_{io} := [y_d \ x_1 \ y_d^{(1)} \ \Phi_1 \ \ldots \ x_{i-1} \ y_d^{(i-1)} \ \Phi_{i-1}]^T$. Choose a constant $\gamma_i > 0$ and a smooth design function $\beta_i: \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id} \to \mathbb{R}$ such that

$$\beta_i(X_{io}, X_{ia}, X_{id}) \geq c_{\beta_i} > 0 \ \forall (X_{io}, X_{ia}, X_{id}) \in \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id}$$

where $c_{\beta_i}$ is a constant. Again, we make the following substitution, by Lemma 1:

$$\sqrt{j_i} \to \delta, \ \beta_i z_i \to \phi, \ \forall (X_{io}, X_{ia}, X_{id}) \in D_{io} \times D_{ia} \times D_{id}$$

Then, $V_i = V_{i+1} + \gamma_i z_i^2$, $V_i: \mathcal{D}_{io} \times \mathcal{D}_{ia} \to \mathbb{R}$, and $x_i: \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id} \to \mathbb{R}$ are smooth and such that

$$\dot{V}_i(X_{io}, X_{ia}, X_{id}, \dot{x}_{i+1}, v)|_{\dot{x}_{i+1} = z_i(X_{io}, X_{ia}, X_{id})}$$

$$= -z_i^2 - \sum_{j=1}^{i} \beta_j z_j^2 - \gamma^2 (v - v_i(X_{io}, X_{ia}, X_{id}))^2 + \gamma^2 v^2$$

$$\forall (X_{io}, X_{ia}, X_{id}) \in \mathcal{D}_{io} \times \mathcal{D}_{id} \ \forall X_{ia} \in \mathcal{D}_{ia} \ \forall v \in \mathcal{D}_w$$

where $v_i: \mathcal{D}_{io} \times \mathcal{D}_{ia} \times \mathcal{D}_{id} \to \mathbb{R}$ is smooth and properly defined.

If $\dot{x}_{i+1}$ had been the actual control input, then we would have used the following virtual control law $\dot{x}_{i+1} = z_i(X_{io}, X_{ia}, X_{id})$ to guarantee the dissipation inequality with supply rate $-z_i^2 - \sum_{j=1}^{i} \beta_j z_j^2 + \gamma^2 v^2$.

This completes this step of backstepping design.

**Step r**: First define the transformed variable $z_r = \dot{x}_r - x_{r-1}(X_{r-1o}, X_{r-1a}, X_{r-1d})$. Similar as step i, we apply Lemma 1 with the following substitutions:

$$X_{ro} := [X_{r-1o}^{(r)} \ X_{r-1a}^{(r)} \ X_{r-1d}^{(r)}]^T \to x_o, \ X_{ra} := \dot{x}_r \to x_o, \ v \to w, \ u \to u$$

$$X_{rd} := [y_d^{(r)} \ \Phi_r \ \dot{x}_{r+1} \ \ldots \ \dot{x}_n \ \Phi_{r+1} \ \ldots \ \Phi_n]^T \to x_d, \ D_{ra} := \mathbb{R} \to D_a$$

$$D_{ro} := \mathcal{D}_{r-1o} \times \mathcal{D}_{r-1a} \times \mathcal{D}_{r-1d} = \{X_{ro}: \dot{\Theta} \in \Theta_o\} \to D_o, \ D_{rd} := \mathcal{D}_w \to D_w$$

$$D_{rd} := \mathbb{R}^{1+n-r+1+n-r} \to D_d, \ \mathcal{D}_{ro} \times \mathcal{D}_{rd} \to D_1, \ V_{r-1} := \sum_{i=1}^{r-1} \gamma_i z_i^2 \to V_o$$

$$v_{r-1} \to \sigma_o, \ x_{r-1} \to \sigma_o, \ z_i^2 + \sum_{j=1}^{i-1} \beta_j z_j^2 \to l_o, \ \gamma \to \gamma$$

$$F_{ro} := [F_{r-1o}^{(r)} \ F_{r-1a} + G_{r-1a} \dot{x}_r \ y_d^{(r)} \ \psi_{r-1} + \Phi_r]^T \to f_o$$

$$H_{ro} := [H_{r-1o}^{(r)} \ H_{r-1a} \ 0 \ c_{n,r-1}' \ A_{211} \ z_{1}^{-1}]^T \to h_o$$

$$F_{ra} := f_r + a_{r+1} \dot{x}_{r+1} + \ldots + a_n \dot{x}_n \to f_a, \ G_{ra} := b_{p0} + A_{2120} \dot{\theta} \to g_a, \ H_{ra} := h_r \to h_a$$

Note that $X_{ro} := [y_d^{(r)} \ \dot{\Sigma} \ s_1 \ x_1 \ y_d^{(1)} \ \Phi_1 \ \ldots \ \dot{x}_{r-1} \ y_d^{(r-1)} \ \Phi_{r-1}]^T$. Choose a constant $\gamma_r > 0$ and a smooth design function $\beta_r: \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$ such that

$$\beta_r(X_{ro}, X_{ra}, X_{rd}) \geq c_{\beta_r} > 0 \ \forall (X_{ro}, X_{ra}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd}$$

where $c_\beta$ is a constant. Again, we make the following substitution, by Lemma 1:

$$\sqrt{r} \to \delta, \quad \beta_r z_r \to \phi, \quad V_r \to V, \quad \mu \to x$$

Then, $V_r = V_{r-1} + \gamma_r z_r^2$, $V_r : \mathcal{D}_r \times \mathcal{D}_r \to \mathbb{R}$, and $\mu : \mathcal{D}_r \times \mathcal{D}_r \times \mathcal{D}_r \to \mathbb{R}$ are smooth.

$$\dot{V}_r(X_{ro}, X_{ra}, X_{rd}, u, v)|_{u=\mu(X_{ro}, X_{ra}, X_{rd})}$$

$$= -z_r^2 - \sum_{j=1}^r \beta_j z_j^2 + \gamma^2 v^2 - \gamma^2 (v - v_r(X_{ro}, X_{ra}, X_{rd}))^2$$

$$\forall (X_{ro}, X_{RD}) \in \mathcal{D}_r \times \mathcal{D}_r \quad \forall X_{ra} \in \mathcal{D}_r \quad \forall v \in \mathcal{D}_w$$

where $v_r : \mathcal{D}_r \times \mathcal{D}_r \times \mathcal{D}_r \to \mathbb{R}$ is smooth and appropriately defined.

According to (A2), we can finish the design of control function

$$u = \mu(X_{ro}, X_{ra}, X_{rd}) = (2\gamma^2 b_0)^{-1}(-j_1(X_{ro}, X_{ra}, X_{rd}) - 2\gamma^2 j_2(X_{ro}, X_{ra}, X_{rd})v_{r-1}$$

$$- \gamma^2 j_2(X_{ro}, X_{ra}, X_{rd})(j_2(X_{ro}, X_{ra}, X_{rd})z_r - \beta_r z_r)$$

where the functions $j_1$ and $j_2$ are defined in (A1a) and (A1b), respectively.

We note that the value function $V = V_r$. This completes the backstepping design procedure.

Next, we consider the design of $U$. The derivative of $V$, when $\dot{x}_c$ is not necessarily 0, is

$$\dot{V}(X_{ro}, X_{ra}, X_{rd}, \dot{x}_c, v) = - (\dot{x}_1 - x_0(X_{io}))^2 - \sum_{j=1}^r \beta_j (\dot{x}_j - x_{j-1}(X_{io}))^2 + \gamma^2 v^2$$

$$- \gamma^2 (v - v_r(X_{ro}, X_{ra}, X_{rd}))^2 + (\dot{x}_r(X_{ro}, X_{ra}, X_{rd}))^2 (\dot{x}_c - x_0(X_{io}))^2$$

$$\forall (X_{ro}, X_{rd}) \in \mathcal{D}_r \times \mathcal{D}_r \quad \forall X_{ra} \in \mathcal{D}_r \quad \forall v \in \mathbb{R} \quad \forall \dot{x}_c \in \mathbb{R}^{\sigma+n}$$

where $\dot{x}_r : \mathcal{D}_r \times \mathcal{D}_r \times \mathcal{D}_r \to \mathbb{R}^{\sigma+n}$ is smooth and appropriately defined: The closed-loop system admits the state vector

$$X := [\theta' \ x' \ X_{ro}' \ X_{ra} \ \Phi_r \ \dot{x}_{r+1} \ \ldots \ \dot{x}_n \ \Phi_{n+1} \ \ldots \ \Phi_{n}]'$$

which belongs to the open set $\mathcal{D} := \{X : \Sigma > 0, \Sigma > 0, \bar{\theta} \in \Theta, \Delta \Sigma = 0\}$. One the basis of the value functions of the estimation design and controller design, we obtain the following value function for the closed-loop adaptive nonlinear system:

$$U = V + W = |\theta - \bar{\theta}|_\Sigma^2 + \gamma^2 |x - \dot{x}|^2 - \Phi(\theta - \bar{\theta})^2 + \gamma^2 |x_{io} - x_{io}|^2$$

$$= \sum_{j=1}^r \gamma_j (\dot{x}_j - x_{j-1})^2$$

where $U : \mathcal{D} \to \mathbb{R}$ is smooth. The derivative of $U$ is given by

$$\dot{U}(X, y_d^{(r)}, \dot{x}, w) = -|x_1 - y_d|^2 - \gamma^4 |x - \dot{x}| - \Phi(\theta - \bar{\theta})^2$$

$$+ |\dot{x}_c + \frac{1}{2} \dot{x}_r(X_{ro}, X_{ra}, X_{rd})|^2 - \frac{1}{4} |\dot{x}_r(X_{ro}, X_{ra}, X_{rd})|^2$$

$$+ 2(\theta - \bar{\theta}) \Phi_s \bar{\theta} - \sum_{j=1}^r \beta_j z_j^2 + \gamma^2 |w|^2 - \gamma^2 |w - w_{opt}(X, y_d^{(r)})|^2$$

$$\forall X \in \mathcal{D} \quad \forall y_d^{(r)} \in \mathbb{R} \quad \forall \dot{x}_c \in \mathbb{R}^{\sigma+n} \quad \forall w \in \mathbb{R}^d$$

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where the worst-case disturbance with respect to the value function $U$ is

$$w_{\text{opt}}(X, Y^{(r)}) = \zeta E' v_r + \gamma^{-2}(I_q - \zeta^2 E'E) \bar{D} \bar{\Sigma}^{-1} (\bar{\xi} - \bar{\zeta}) + \zeta^2 E C (\bar{x} - x)$$

and $w_{\text{opt}}: \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}^q$ is smooth.

The design of $\bar{\zeta}$ is based on two considerations: (1) to guarantee $\|0 \gamma^{-2} \Pi \bar{Q} \bar{\xi}_c\|$ is bounded, which is applied later in the proof of Theorem 1; (2) to guarantee the terms $|\bar{\zeta} + \frac{1}{2} \bar{s} \bar{\zeta}^2 - \frac{1}{4} |\bar{s}|^2 + \bar{s}^2 \bar{C} (\bar{x} - x)$ is nonpositive. Then, we choose $\bar{\zeta}$ to be

$$\bar{\zeta} := \mu_0(X_{ro}, X_{ra}, X_{rd}) = -\frac{1}{2} \varepsilon_0(X_{ro}, X_{ra}, X_{rd}) \bar{s} \bar{s} + (X_{ro}, X_{ra}, X_{rd})$$

where

$$\varepsilon_0(X_{ro}, X_{ra}, X_{rd}) := \frac{|k_c|}{\sqrt{k_c^2 + (0 \gamma^{-2} \Pi \bar{Q} (\Phi, s \Sigma) \bar{s} (X_{ro}, X_{ra}, X_{rd})|^2 + 1}}$$

and $k_c \in \mathbb{R}$ is a constant. Clearly, $\mu_0: \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \rightarrow \mathbb{R}^{n+\sigma}$ is a smooth function. We can simply take $k_c = 0$.

Remark 4
With $\varepsilon = 1$, the resulting reduced-order adaptive control system involves $4n + \sigma$ integrators, which includes the dynamics of $x$, $\bar{x}$, $\bar{\eta}$, and $\bar{\lambda}$. The reduced-order control structure has $n$ integrators less than that of the full-order control structure in [32], where $5n + \sigma$ integrators are involved with $\varepsilon = 1$. In the full-order controller design, an $n$-dimensional dynamics $\eta_{\bar{d}}$ is generated at step 0 of backstepping design for $\eta$ to track in order to stabilize the $\eta$ dynamics. In this paper, we start backstepping from step 1, without first stabilizing $\eta$ dynamics. Then, there is no need to generate new dynamics for $\eta$ to track; the dynamic order of the controller is thereby reduced by $n$. The boundedness of $\eta$ will be alternatively proved via the existing state variable $\bar{x} - \Phi \bar{\eta}$, which admits the desired structure we can utilize for stability analysis.

In addition, the proposed reduced-order control scheme can further simplify the structure of the reduced-order controller obtained in [34] by 1 or 2 integrators.

The stability of the closed-loop system cannot be deduced directly from the value function $U$, which is not a positive-definite function. Next, we will turn to study the robustness and tracking properties of the closed-loop system.

6. MAIN RESULT

The robustness and tracking properties of the closed-loop system with the reduced-order controller are made precise in the following theorem.

Theorem 1
Consider the robust adaptive control problem formulated in Section 2, with Assumptions 1–5 holding. Then, the robust adaptive controller $\mu$ defined by (31) or (33), with $\bar{\zeta}_c$ given by (37), achieves strong robustness properties for the closed-loop system.
1. Given \( c_w \geq 0 \) and \( c_d \geq 0 \), there exists a constant \( c_e \geq 0 \) and a compact set \( \Theta_e \subset \Theta_o \), such that, for any uncertainty \( (x_0, \theta, \dot{w}_{[0, \infty)}, Y_{d0}, Y_{d1}[0, \infty)) \in \tilde{W}^c \) with

\[
|x_0| \leq c_w, \quad |\dot{w}(t)| \leq c_w, \quad |Y_{d(t)}| \leq c_d \quad \forall t \in [0, \infty)
\]

all closed-loop state variables \( x, \dot{x}, \theta, \Sigma, s, \) and \( \Phi \) are bounded as follows: \( \forall t \in [0, \infty) \)

\[
|x(t)| \leq c_e, \quad |\dot{x}(t)| \leq c_e, \quad \dot{\theta}(t) \in \Theta_e, \quad |\Phi(t)| \leq c_e
\]

\[
K^{-1}_e I_\Sigma \leq \Sigma(t) \leq \gamma^{-2} Q^{-1}_0, \quad \gamma^2 \text{Tr} \,(Q_0) \leq s(t) \leq K_c
\]

to there is a compact set \( S \subset \mathcal{D} \) such that \( X(t) \in S, \forall t \in [0, \infty) \). Hence, there exists a constant \( c_u \geq 0 \) such that \( |u(t)| \leq c_u, |\dot{z}(t)| \leq c_u, |\eta(t)| \leq c_u, \) and \( |\dot{\lambda}(t)| \leq c_u, \forall t \in [0, \infty) \).

2. The controller \( \mu \) belongs to \( \mathcal{M}_u \) and achieves disturbance attenuation level \( \gamma \) for any uncertainty \( (x_0, \theta, \dot{w}_{[0, \infty)}, Y_{d0}, Y_{d1}[0, \infty)) \in \tilde{W}^c \).

3. For any uncertainty \( (x_0, \theta, \dot{w}_{[0, \infty)}, Y_{d0}, Y_{d1}[0, \infty)) \in \tilde{W}^c \) with \( \dot{w}_{[0, \infty)} \in L_2 \cap L_\infty \) and \( Y_{d1}[0, \infty) \in L_\infty \), the output \( x_1 \) asymptotically tracks the reference trajectory \( y_d \), i.e.

\[
\lim_{t \to \infty} (x_1(t) - y_d(t)) = 0
\]

**Proof**

For the first statement, fix any uncertainty \( (x_0, \theta, \dot{w}_{[0, \infty)}, Y_{d0}, Y_{d1}[0, \infty)) \in \tilde{W}^c \) such that \( |x_0| \leq c_w, |\dot{w}(t)| \leq c_w, |Y_{d(t)}| \leq c_d \), \( \forall t \in [0, \infty) \), for some \( c_w \geq 0 \) and \( c_d \geq 0 \). Consider the maximal length interval \( [0, T_f] \) such that the differential equation for the closed-loop system admits a solution that lies in \( \mathcal{D} \) on this interval. It will be shown that the maximal length of the interval \( T_f \) is always \( \infty \).

Under the definition of \( \varepsilon(t) = \text{Tr}((\Sigma(t))^{-1})/K_c \), Remark 2 states that the covariance matrix \( \Sigma \) and then the signal \( s \) are bounded as desired. If \( \varepsilon(t) = 1, \Sigma, \) and \( s \), are constants and therefore bounded as desired, too.

Define \( \eta_c = \dot{x} - \Phi \dot{\theta} \). Introduce the new vector \( X_e := (\dot{\theta}, \eta_c, z_1, \ldots, z_r)^T \), where \( X_e : [0, T_f) \to \mathcal{D}_e := \{X_e : \dot{\theta} \in \Theta_e\} \). Then, the value function \( U \) can be explained as \( U = \tilde{U}(t, X_e(t)) \), where \( \tilde{U} : [0, T_f) \times \mathbb{R}^{n+\sigma+r} \to \mathbb{R} \). Since \( 0 \leq \varepsilon_0 < 1, (\varepsilon_0^2 - \varepsilon_0)|\dot{z}_i|^2/\tilde{Q}/2 < 0 \), we have that the derivative of \( \tilde{U} \) satisfies the inequality:

\[
\dot{\tilde{U}} \leq -\gamma^2/2 |\eta_c|^2 \Pi^{-1} \Delta \Pi^{-1} + 2(\theta - \dot{\theta})' P_r (\dot{\theta}) - \sum_{j=1}^r c_\beta_j z_j^2 + \gamma^2 \|\tilde{M}\|^2 c_w^2
\]  \hspace{1cm} (38)

The right-hand side of inequality (38) approaches negative infinity as \( X_e \) approaches the boundary of \( \mathcal{D}_e \). Then there exists a compact set \( \Omega_1(e_w) \subset \mathcal{D}_e \) such that, \( \forall t \in [0, T_f) \), if \( X_e(t) \in \mathbb{R}^{n+\sigma+r} \setminus \Omega_1(e_w) \) then \( X_e(t) \in \mathcal{D}_e \setminus \Omega_1(e_w) \) and \( \tilde{U} < 0 \). Define

\[
U_M(X_e) := K_c |\theta - \dot{\theta}|^2 + \gamma^2 |\eta_c|^2 \Pi^{-1} + \sum_{j=1}^r \gamma_j z_j^2, \quad U_m(X_e) := \gamma^2 |\theta - \dot{\theta}|^2 \tilde{Q}_0 + \gamma^2 |\eta_c|^2 \Pi^{-1} + \sum_{j=1}^r \gamma_j z_j^2
\]

Then, we have \( U_m(X_e) \leq \tilde{U}(t, X_e) \leq U_M(X_e), \forall t \in [0, T_f), X_e \in \mathbb{R}^{n+\sigma+r}. \) Define sets \( S_{1e} := \{X_e \in \mathbb{R}^{n+\sigma+r} | U_m(X_e) \leq \varepsilon \}, \varepsilon \in \mathbb{R} \). Then, they are compact sets, since \( U_m(X_e) \) is continuous and radially unbounded. By Lemma 5, we can conclude that there is a compact set \( S_{1e} \subset \mathbb{R}^{n+\sigma+r} \).
such that $X_r(t) \in S_{1c_1}$, $\forall t \in [0, T_f)$, for some $c_1 \in \mathbb{R}$. Therefore, we conclude that the components of $X_r$: $\tilde{\theta}$, $\eta_c$, and $z_1, \ldots, z_r$ are bounded.

Considering the state $\eta_c$, whose derivative is given by

$$\dot{\eta}_c = A_f \eta + y^2 - 2 \Pi C' (yd - y) + Dv - (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^2)) Ew - \frac{1}{2} [0 \ y^2 - \Pi] \tilde{Q} \tilde{e}_0 \tilde{z}_r$$

we express $\eta_c$ in two parts $\eta_c = \eta_{cy} + \eta_{cyd}$, where

$$\dot{\eta}_{cy} = A_f \eta_{cy} - y^2 \Pi C' y$$

$$\dot{\eta}_{cyd} = A_f \eta_{cyd} + y^2 - 2 \Pi C' y d - (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^2)) Ew + Dv - \frac{1}{2} [0 \ y^2 - \Pi] \tilde{Q} \tilde{e}_0 \tilde{z}_r$$

(39a) (39b)

Note that $A_f$ is Hurwitz, $y_d$ and $w$ are bounded, $\frac{1}{2} [0 \ y^2 - \Pi] \tilde{Q} \tilde{e}_0 \tilde{z}_r$ is bounded via the design of $\tilde{z}_r$. Therefore, $\eta_{cyd}$ is bounded, so is $\eta_{cy}$ by the boundedness of $\eta_c$ and $\eta_{cyd}$.

By dynamics (39b) of $\eta_{cy}$, there is a particular linear combination of the elements of $\eta_{cy}$, denoted by $\eta_L$, which is bounded, SMP and has relative degree 1 with respect to the input $y$. By Lemma 2, the $\eta_{cy}$ dynamics with input $y$ and output $\eta_L$ may serve as a reference system mentioned in Lemma 3. Applying Lemma 4, we know that $\eta_L$ is SMP and has relative degree $r + 1$ with respect to the input $u$; and the composite system with input $u$ and output $\eta_L$ may also serve as a reference system.

To analyze the boundedness of $\eta$, we observe that the relative degree for each element of $\eta$ is at least 1 with respect to the input $y$ and is the output of a stable linear system. By Lemma 3, we have that $\eta$ is bounded, where the reference system has output $\eta_L$ and input $y$.

Next consider the boundedness of $\Phi$, which can be decomposed into $y$ dependent part and $u$ dependent part as follows:

$$\Phi = \Phi_y + \Phi_u$$

$$\Phi_y = [A_f^{-1} \eta \ldots A_f \eta] M_f^{-1} \tilde{A}_{211} = [T_1 \eta \ldots T_n \eta]'$$

$$\Phi_u = A_f \Phi_u + u \tilde{A}_{212}, \quad \Phi_u(0) = \Phi_{u0}$$

(40a) (40b) (40c)

where $\Phi_{u0}$ is the matrix such that $\Phi_y(0) + \Phi_{u0} = \Phi_0$. Clearly, $\Phi_y$ is bounded since $\eta$ is bounded. We can conclude the boundedness of $\Phi$ if $\Phi_u$ is proved to be bounded. Express $\Phi_u$ in terms of its row vectors $[\Phi_{u1} \ldots \Phi_{un}]'$. We consider the boundedness of $\Phi_{u1}$ in two steps. Define

$$\lambda = A_f \lambda + e_{n,t} u, \quad \lambda(0) = 0_{n \times 1}$$

$$\Phi_{u1} = A_f \Phi_{u1} + u \tilde{A}_{2120}, \quad \Phi_{u1}(0) = \Phi_{u0}$$

(41a) (41b)

Then, we have $\Phi_u = \Phi_{u1} + \lambda \tilde{A}_{2120}$. We express $\lambda$ in terms of its elements $[\lambda_{b1} \ldots \lambda_{bn}]'$; and $\Phi_{u1}$ in terms of its row vectors $[\Phi_{u1} \ldots \Phi_{un}]'$.

The relative degree for each element of $\Phi_{u1}$ is at least $r + 1$ with respect to the input $u$ and is the output of a stable linear system. By Lemma 3, this yields that $\Phi_{u1}$ is bounded, where the reference system has output $\eta_L$ and input $u$ and $\dot{u}$.

By the boundedness of $\tilde{z}_1 = \Phi \dot{\eta}, \eta, \Phi_{u1},$ and $\dot{\eta}$, we have that the signal $\tilde{x}_1 = \dot{\lambda}_{b1} \tilde{A}_{2120} \tilde{\theta}$ is bounded, so is $\tilde{x}_1$ in view of the boundedness of $z_1$ and $y_d$.

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Notice the signal $\dot{x}_1 - \lambda b_1 \ddot{A}_{2120} = (x_1 - \dot{\lambda}_b b_0) - (\dot{x}_1 - \lambda b_1 (b_p + \ddot{A}_{2120} \ddot{\theta}))$. We analyze the item $x_1 - \dot{\lambda}_b b_0$ first, which can be generated by

$$
\dot{x} - b_0 \ddot{x}_b = A_f (x - b_0 \dot{\lambda}_b) + \left[ \begin{array}{c} 0_{r \times 1} \\ \ddot{A}_{212r}, \theta \end{array} \right] u + (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^{-2})) (y - E \dot{M} \ddot{w})
$$

$$
+ [0_{1 \times r}, b_{p1}, \ldots, b_{p, n-r}]' u + \ddot{A}_{2111} \theta y + D \dot{M} \ddot{w}
$$

(42a)

Recalling the way to decompose $\Phi$, we similarly decompose the signal $x_1 - \dot{\lambda}_b b_0$ into $y$ dependent part and $u$ dependent part:

$$
\dot{x}_u = A_f (x_u) + \left[ \begin{array}{c} \theta_r \times 1 \\ \ddot{A}_{212r}, \theta \end{array} \right] u + [0_{1 \times r}, b_{p1}, \ldots, b_{p, n-r}]' u, \quad x_u(0) = x_0
$$

(43a)

$$
x_{u1} = C x_u
$$

(43b)

$$
\dot{x}_y = A_f (x_y) + (\zeta^2 L + \Pi C' (\zeta^2 - \gamma^{-2})) (y - E \dot{M} \ddot{w}) + \ddot{A}_{2111} \theta y + D \dot{M} \ddot{w}, \quad x_y(0) = 0_{n \times 1}
$$

(43c)

$$
x_{y1} = C x_y
$$

(43d)

$$
x_1 - b_0 \dot{\lambda}_b = x_{u1} + x_{y1}
$$

(43e)

The signal $x_{u1}$ has relative degree at least $r + 1$ with respect to $u$. It is bounded by Lemma 3, where the reference system has input $u$ and output $\eta_L$. The signal $x_{y1}$ has relative degree of at least 1 with respect to $y$. It is bounded by Lemma 3, where the reference system has input $y$ and output $\eta_L$. Then, $x_1 - b_0 \dot{\lambda}_b$ is bounded on $[0, T_f]$. It can be further concluded that $\dot{x}_1 - \dot{\lambda}_b (b_p + \ddot{A}_{2120} \ddot{\theta})$ is bounded.

Since $\dot{x}_1$ is bounded, and $\ddot{b}_0 := b_p + \ddot{A}_{2120} \ddot{\theta}$ is bounded away from 0 by the fact that $\ddot{\theta}(t) \in \Theta_o, \forall t \in [0, T_f]$, we have the boundedness of the signal $\dot{\lambda}_b$. This further implies the boundedness of the signals $\Phi_{u1}$ and $\Phi_1$ and the boundedness of $x_1$ and $y$ because of the boundedness of $x_1 - b_0 \dot{\lambda}_b$ and $\ddot{w}$.

To show the existence of a compact set $\Theta_c \subset \Theta_o$ such that $\ddot{\theta}(t) \in \Theta_c, \forall t \in [0, T_f]$, we define the function $\hat{\Upsilon} := U + P (\ddot{\theta})(\rho_o - P (\ddot{\theta}))^{-1}$, where $\hat{\Upsilon}$ can be explained as $\hat{\Upsilon} (X(t)) := \hat{\Upsilon} (t, X_c(t))$, where $\hat{\Upsilon} : [0, T_f) \times \mathcal{D}_c \rightarrow \mathbb{R}$ is 6. The time derivative of $\hat{\Upsilon}$ is given by

$$
\hat{\Upsilon} \dot{t} = \dot{U} + \rho_o (\rho_o - P (\ddot{\theta}))^{-2} \frac{\partial P}{\partial \theta} (\ddot{\theta}) \ddot{\theta}
$$

$$
\leq -\gamma^4 / 2 |x - \ddot{x} - \Phi (\theta - \ddot{\theta})|^2 \Pi^{-1} \Delta \Pi^{-1} + 2(\theta - \ddot{\theta})' P_r (\ddot{\theta}) - \sum_{j=1}^{r} \frac{\partial^2 P}{\partial \theta^2} (\ddot{\theta}) c_j z_j^2
$$

$$
- \frac{\rho_o}{K_c} \ddot{P}_r (\ddot{\theta}) (\rho_o - P (\ddot{\theta}))^{-2} \left( \frac{\partial P}{\partial \theta} (\ddot{\theta}) \right)' + (\rho_o - P (\ddot{\theta}))^{-4} \left( \frac{\partial P}{\partial \theta} (\ddot{\theta}) \right)'^2 c + c
$$

(44)

for some constant $c > 0$. This inequality follows from the boundedness of $y_d, x_1, \dot{x}_1, \Phi$, and $\ddot{w}$. The right-hand side of the inequality tends to $-\infty$ as $X_c$ approaches the boundary of $\mathcal{D}_c$. Therefore, there
exist a compact set $\Omega_2 \subset \mathcal{D}_e$, such that $\forall X_e(t) \in \mathcal{D}_e \setminus \Omega_2$ implies $\hat{Y} < 0$. Note also that, $\forall (t, X_e) \in [0, T_f) \times \mathcal{D}_e$, it yields

$$U_m(X_e) + P(\bar{\theta})(\rho_o - P(\bar{\theta}))^{-1} \leq \bar{Y}(t, X_e) \leq U_M(X_e) + P(\bar{\theta})(\rho_o - P(\bar{\theta}))^{-1}$$

Define sets $S_{2c} \coloneqq \{ X_e \in \mathcal{D}_e | U_m(X_e) + P(\bar{\theta})(\rho_o - P(\bar{\theta}))^{-1} \leq \bar{z}, \bar{z} \in \mathbb{R} \}$. Then, they are compact. By Lemma 5, we can conclude that there exists a compact set $S_{2c} \subset \mathcal{D}_e$ such that $X_e(t) \in S_{2c}$, $\forall t \in [0, T_f)$, for some $c_2 \in \mathbb{R}$. Therefore, there exists a compact set $\Theta_c \subset \Theta_o$ such that $\bar{\theta}(t) \in \Theta_c$, $\forall t \in [0, T_f)$.

Further proof of the first statement will only consider the case of $r \geq 2$. The proof for the case of $r = 1$ can be automatically deduced from the case $r \geq 2$.

Now, consider signal $x_1$ instead of $\eta_o$. This signal is bounded on $[0, T_f)$ and is minimum phase, with relative degree $r$ with respect to input $u$. By a similar bounding analysis as the one above, we can deduce the boundedness of the signals $\Phi u, \dot{x}_2 - \lambda_{b_2} \dot{b}_0, \ddot{x}_2, x_2 - b_0 \dot{\lambda}_{b_2}, \dot{\lambda}_{b_2}$, and $x_2$. Now, take a linear combination of $x_1$ and $x_2$ that is minimum phase and has relative degree $r-1$ with respect to $u$. Instead of $x_1$, we can inductively conclude the boundedness of $\ddot{x}_3, \dot{x}_3, \lambda_{b_3}, \ldots, \dot{x}_r, x_r, \lambda_{b_r}$. Then the state $x$ is bounded, since it can be viewed as a stably filtered output signal of $u$ and $y$.

Since $\eta$ and $\lambda$ are some stably filtered output signals of $u$ and $y$, they are bounded. Then, the signal $\Phi$ is bounded. Therefore, $\tilde{x}$ is bounded in view of the boundedness of $\dot{x} - \Phi \dot{\theta}$.

Hence, we conclude that there exists a compact set $\mathcal{F} \subset \mathcal{D}$ such that $X(t) \in \mathcal{F}$. Therefore, we can conclude that $T_f = +\infty$ by Theorem 3.3 in [41]. This further implies that the control input $u$ and $\hat{\xi}$ are bounded. Thus, this completes the proof of statement 1.

Next, we prove the second statement. $\forall t_f \in (0, \infty) \subseteq \mathbb{R}$, we know for any uncertainty $(x_0, \theta, \dot{w}(0, \theta), \dot{y}_d(0, \infty)) \in \mathcal{W}$, there exists constants $c_w \geq 0$ and $c_d \geq 0$ such that $|x_0| \leq c_w$, $|\dot{w}(t)| \leq c_w$, and $|Y_d(t)| \leq c_d$, $\forall t \in [0, t_f]$. By the first statement and the causality of the closed-loop system, there exists a solution $X : [0, t_f] \rightarrow \mathcal{D}$ for the closed-loop system. Therefore, the closed-loop system admits a unique solution on $[0, \infty)$, which then implies that the proposed adaptive control law belongs to $\mathcal{M}_a$.

Now we choose $l_0 = V(X_{ro}(0), X_{ra}(0))$ and choose $l$ as

$$l(t, \theta, x, y[0, t], Y_d[0, t]) = \gamma r |x - \tilde{x} - \Phi(\theta - \bar{\theta})|^2 + \varepsilon |\dot{x}_r|^2 + 2(\theta - \bar{\theta})^T \dot{P}_r(\bar{\theta}) - \varepsilon \bar{\theta}^2 |\dot{x}_r|^2/2 + \sum_{j=1}^{r} \beta_j z_j^2$$

The function $l$ is nonnegative along the closed-loop solution. Then, we obtain

$$J_{\gamma t_f} = J_{\gamma t_f} + \int_0^{t_f} \dot{U}(X(\tau), y_d^{(r)}(\tau), w(\tau)) d\tau + U(X(0)) - U(X(t_f)) \leq -U(X(t_f)) \leq 0$$

This shows that the controller $\mu$, along with the choice of $\hat{\xi}$, achieves the disturbance attenuation level $\gamma$ as described in Definition 1. Thus, the second statement is established.
We continue with the proof of the third statement. For any uncertainty \((x_0, \theta, \dot{w}_d, Y_{d[0,\infty])} \in \mathcal{Y}\) with \(\dot{w}_d \in \mathcal{L}_2 \cap \mathcal{L}_\infty\) and \(Y_{d[0,\infty)} \in \mathcal{L}_\infty\), we have
\[
\int_0^\infty |x_1(t) - y_d(t)|^2 dt \leq U(X(0)) + \int_0^\infty \gamma^2 |\dot{M} \dot{w}(t)|^2 dt < +\infty
\]
by inequality (35) and the second statement. It implies that \((x_1 - y_d) \in \mathcal{L}_2\) on the interval \([0, \infty)\).

From the first statement, we have that \(\dot{x}_1 - \dot{y}_d \in \mathcal{L}_\infty\) on the interval \([0, \infty)\). Therefore,
\[
\lim_{t \to \infty} (x_1(t) - y_d(t)) = 0
\]
This completes the proof of the theorem. \(\Box\)

7. EXAMPLE

In this section, an example is presented for performance illustration of the reduced-order controller. We consider a first-order system:
\[
\begin{align*}
\dot{x}_1 &= x_1 + u + w_d + [0.1, 0] \dot{w} \\
y &= \dot{x}_1 + [0, 0.1] \dot{w}
\end{align*}
\]
where the exogenous disturbances include two components: a dominant sinusoidal waveform \(w_d\) and an arbitrary waveform \(\dot{w}\) of smaller magnitude. The sinusoidal component \(w_d\) has unknown magnitude, frequency, and phase. We can model it as a second-order linear system:
\[
\begin{align*}
\dot{x}_2 &= \dot{x}_3, \\
\dot{x}_3 &= \theta \dot{x}_2, \text{ and } w_d = \dot{x}_2.
\end{align*}
\]
This disturbance model is then integrated into the system model (46) and we have
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & \theta & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u +
\begin{bmatrix}
0.1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \dot{w}
\]
\[
y = [1, 0, 0] \dot{x}_1 + [0, 0.1] \dot{w}
\]
For simulation purposes, the initial condition for the state \(\dot{x}\) and the true value of the parameter \(\theta\) are set to be: \(x_1(0) = 1, \dot{x}_2(0) = 1, \dot{x}_3(0) = 2, \text{ and } \theta = -4\). Then, the sinusoidal disturbance \(w_d(t)\) is \(\sqrt{2} \sin(2t + \pi/4), \forall t \geq 0\).

The design model for the adaptive controller is obtained as
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} x +
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
-1 & 0
\end{bmatrix} u +
\begin{bmatrix}
0.1 & 0 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0
\end{bmatrix} \dot{w}
\]
\[
y = [1, 0, 0] x + [0, 0, 0.1] \dot{w}
\]
via the state and disturbance transformation

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  -\theta & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \dot{x} \\
  \dot{w} \\
  \dot{w}
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & -\theta & -\theta \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  \theta
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  \dot{x} \\
  \dot{y} \\
  \dot{\theta}
\end{bmatrix}
\]

The reference trajectory is generated by the following differential equation:

\[
\dot{x}_d = -x_d + d
\]
\[
y_d = x_d
\]

where \( d \) is the command signal. The initial state \( x_d(0) = 0 \).

The projection function \( P(\theta) \) is chosen to be \( P(\theta) = 0.01(\theta + 10)^2 \). The initial states for the controller are set to be \( \tilde{\theta}_0 = -1 \), \( \tilde{x}_0 = [1 \ 0 \ 0]^T \), and \( Q_0 = 1 \). The desired disturbance attenuation level is \( \gamma = 0.2 \). The constant \( k_c \) in the expression of \( \xi_c \) is chosen to be 0; then we have \( \xi_c = 0 \).

The other parameters are

\[
p_n = [0 \ 0 \ 1]^T, \quad \Delta_1 = I_3, \quad \beta_\Delta = 0, \quad K_c = 0.1 \gamma_2 Q_0, \quad \rho_o = 1.5
\]
\[
\gamma_1 = 0.5, \quad \beta_1 = 0.5, \quad \varepsilon = K_c^{-1} s_\Sigma
\]

The matrix \( \Pi \) is obtained from Equation (12) and the matrix \( A_f \) is calculated as follows:

\[
\Pi = 
\begin{bmatrix}
  0.0618 & 0.0563 & 0.0258 \\
  0.0563 & 0.1786 & 0.0938 \\
  0.0258 & 0.0938 & 0.1090
\end{bmatrix}, \quad A_f = 
\begin{bmatrix}
  -3.6321 & 1 & 0 \\
  -4.2210 & 0 & 1 \\
  -1.9365 & 0 & 0
\end{bmatrix}
\]

By selecting \( \varepsilon = K_c^{-1} s_\Sigma \), the estimator dynamics are given below

\[
\dot{\Sigma} = -3(1 - 10 s_\Sigma) \Sigma \Phi_1^t \Phi_1 \Sigma, \quad \Sigma(0) = 25
\]
\[
\dot{\xi}_\Sigma = 3(1 - 10 s_\Sigma) \Phi_1 \Phi_1^t, \quad s_\xi(0) = 0.04
\]
\[
\dot{\gamma} = A_f \gamma + p_n \gamma, \quad \gamma(0) = 0_{3 \times 1}
\]
\[
\dot{\lambda} = A_f \lambda + p_n u, \quad \lambda(0) = 0_{3 \times 1}
\]
\[
\Phi = 
\begin{bmatrix}
  \Phi_1 \\
  \Phi_2 \\
  \Phi_3
\end{bmatrix} =
\begin{bmatrix}
  -4.6321 \eta_1 + \eta_2 - \dot{\lambda}_1 \\
  -4.2210 \eta_1 - \eta_2 + \eta_3 - \dot{\lambda}_2 \\
  -1.9365 \eta_1 - \eta_3 - \dot{\lambda}_3
\end{bmatrix}
\]
\[
\dot{\tilde{\theta}} = -\Sigma P_r(\tilde{\theta}) - \Sigma \Phi_1 (y_d - \tilde{x}_1) + 4 \Sigma \Phi_1 (y - \tilde{x}_1)
\]
\[
\dot{\tilde{x}}_1 = -\Phi_1 \Sigma P_r(\tilde{\theta}) + \tilde{x}_1 + \tilde{x}_2 + u - (1.5440 + \Sigma \Phi_1^2)(y_d - \tilde{x}_1) + (6.1761 + 4 \Sigma \Phi_1^2)(y - \tilde{x}_1)
\]
\[
\dot{\tilde{x}}_2 = -\Phi_2 \Sigma P_r(\tilde{\theta}) + \tilde{\theta} y + \tilde{x}_3 - (1.4070 + \Sigma \Phi_1 \Phi_2)(y_d - \tilde{x}_1) + (5.6280 + 4 \Sigma \Phi_1 \Phi_2)(y - \tilde{x}_1)
\]
\[
\dot{\tilde{x}}_3 = -\Phi_3 \Sigma P_r(\tilde{\theta}) - \tilde{\theta} u - (0.64550 + \Sigma \Phi_1 \Phi_3)(y_d - \tilde{x}_1) + (2.5820 + 4 \Sigma \Phi_1 \Phi_2)(y - \tilde{x}_1)
\]
The reduced-order control law is obtained as

\[
\mu = -\dot{x}_1 - \dot{x}_2 + \Phi_1 \Sigma P_r (\dot{\theta}) + \dot{y}_d - (3.0440 + \Phi_1^2 \Sigma)(\dot{x}_1 - y_d) \\
- (0.30881 + 0.2\Phi_1^2 \Sigma)(7.7201 + 5\Phi_1^2 \Sigma)(\dot{x}_1 - y_d)
\]  

(47)

We present the simulation results in Figures 1 and 2. The first set of simulation (Figure 1) is done by setting the reference trajectory and the arbitrary exogenous disturbance to be zeros. Left-side column plots show the system response with the reduced-order controller. Right-side column plots show the system response with the full-order controller. For both types of controllers, the tracking errors converge to zero and the parameter estimates converge to the true value, which is consistent with our theoretical findings. The control magnitudes are bounded by 7. The transient response for the reduced-order controller is worse, which may be due to the disappearance of the negative drift relating to \(\dot{\theta}\) in the derivative of the closed-loop value function. However, from long-term observation (not showing here due to paper length), we see that the convergence rates of the tracking error and the parameter estimation error are faster for the reduced-order controller than those for the full-order controller. We also find that larger value of \(k_c\) leads to better transient response, since \(\beta\) is larger accordingly and more negative drift is generated for the stability of the closed-loop system. Different values of \(k_c\) and \(\rho_o\) have not been seen much difference in simulation. However, nonzero \(k_c\) makes \(\xi_c\) more complex and increases computation burden. The farther \(\rho_o\) is from 1, the wider the bound of \(\dot{\theta}\) allowed.

Another set of simulation (Figure 2) is made by setting the command signal to be \(d(t) = 2.5 \sin(2.8t)\) and the disturbance input to be

\[
\dot{w}(t) = \begin{bmatrix}
0.3 \sin \left(3t + \frac{\pi}{4}\right) \\
\text{Band-limited white noise with power 0.1, seed 1000, and sampling period 1 s}
\end{bmatrix}
\]  

(48)

On the basis of Figure 2, we observe that the desired attenuation level is achieved for both of the controllers. The parameter estimates asymptotically oscillate around the true value. The control magnitudes are bounded by 8 for both of the controllers. The transient response for the full-order controller is better, but both of the controllers have comparable steady-state performance from longer-time simulation observation.

Additionally, to show the performance improvement of robust adaptive control design proposed in this work over nonadaptive robust control design, an \(H_\infty\) controller is designed to attenuate the disturbance input \(w_d + [0.1\ 0]\dot{w}\) in system (46). Define \(\bar{w} = [w_d + 0.1\dot{w}_1\ \dot{w}_2]\). Then, system (46) is rewritten as

\[
\dot{x}_1 = \dot{x}_1 + u + [1\ 0]\bar{w} \\
y = \dot{x}_1 + [0\ 0.1]\bar{w}
\]

The cost function is introduced as

\[
J_\gamma = \int_0^\infty \left( (\dot{x}_1 - y_d)^2 + 0.01(u - u_d)^2 - \gamma^2 |\bar{w}|^2 \right) dt
\]

where \(u_d = -y_d + \dot{y}_d\) is the nominal control input for \(\dot{x}_1\) tracking \(y_d\) when system is free of disturbance. Then, the optimal performance level \(\gamma^*_H\) can be calculated via \(H_\infty\) theory as \(\gamma^*_H = 0.152\).
Figure 1. System response under $d(t) = 0$ and no arbitrarily varying disturbance for reduced-order controller (left-column plots) and full-order controller (right-column plots): (a), (b) tracking error; (c), (d) parameter estimate; and (e), (f) control input.
Figure 2. System response under $d(t) = 2.5 \sin(2.8t)$ and arbitrarily varying disturbance $\dot{w}$ for reduced-order controller (left-column plots) and full-order controller (right-column plots): (a), (b) tracking error; (c), (d) parameter estimate; and (e), (f) control input.
Then we still select the same desired attenuation level $\gamma_0 = 0.2 > \gamma_{\infty}$, as before. We have the $H_\infty$ controller

$$u = -22.329 x_H - y_d + \dot{y}_d$$

$$\dot{x}_H = -31.047 x_H + 12.957 y - 12.957 y_d, \quad x_H(0) = 1.0$$

We first set arbitrary disturbance input $\dot{w}_1$ and $\dot{w}_2$ to be zeros; then the system is merely affected by sinusoidal disturbance input $w_d$. The simulation results are shown in Figure 3. We observe that the steady-state tracking error is a sinusoidal waveform, not like in Figure 1(a), where the influence of sinusoidal signals is completely attenuated. Furthermore, control effort in Figure 3 is bounded by 25, much larger than that in Figure 1(e). Now if we set $\dot{w}_1$ and $\dot{w}_2$ to have the same values as (48), the simulation results are depicted in Figure 4. It shows

![Figure 3](image1.png)

Figure 3. System response under $d(t) = 0$ and no arbitrarily varying disturbance for $H_\infty$ controller: (a) tracking error and (b) control input.

![Figure 4](image2.png)

Figure 4. System response under $d(t) = 2.5 \sin(2.8 t)$ and arbitrarily varying disturbance for $H_\infty$ controller: (a) tracking error and (b) control input.
that the control law is still bounded by 25, much larger than the bound of the reduced-order control law in Figure 2(e). The steady-state tracking error has a magnitude of 0.27. However, the tracking error in the adaptive case only has magnitude of about 0.16 during steady-state status (Figure 2(a)). In view of the above simulation comparison, we have that the adaptive control design in this paper has smoother control and smaller steady-state tracking errors compared with the non-adaptive one.

8. EXTENSION TO SISO LINEAR SYSTEMS WITH PARTLY MEASURED DISTURBANCES

Order-reduction method proposed in this paper can also be applied to reduce the order of the adaptive controllers for a class of SISO linear systems with partly measured disturbances [35], where the systems under consideration allow the following representation:

\[
\dot{x} = \hat{A}x + \hat{B}u + \hat{D}w + \hat{D}\tilde{w}, \quad x(0) = \hat{x}_0
\]

\[
y = \hat{C}x + \hat{E}\tilde{w}
\]  

where \(\dot{x}\) is the \(n\)-dimensional state vector; \(n \in \mathbb{N}\); \(u\) is the scalar control input; \(y\) is the scalar system output; all signals in the system are assumed to be continuous, i.e. in the space \(\mathcal{C}\); and the matrices \(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{D}\), and \(\hat{E}\) are generally unknown or partially unknown. Different from system (1), the disturbance input in (49) includes two parts: \(\tilde{w}\) is the \(\tilde{q}\)-dimensional unmeasured disturbance input, \(\tilde{q} \in \mathbb{N}\); and \(\tilde{w}\) is the \(\tilde{q}\)-dimensional measured disturbance input vector, \(\tilde{q} \in \mathbb{N}\).

A detailed derivation of the reduced-order adaptive controllers for system (49) will not be presented here in view of the page limit, since the design procedure is essentially similar as that in this paper. We will present a simulation example below to compare the performance of the reduced-order controller obtained using the approach in this paper and the full-order controller (derived in [35]). For comparison purpose, we still use the circuit example provided by Zeng and Pan [35].

Consider a circuit shown in Figure 5. The resistance \(R\) is 1\(\Omega\). The capacitor \(C\) and the inductor \(L\) are linear and time invariant. We know \(L = 1H\). \(C\) is unknown parameter with true value \(1(F)\) for simulation purpose. \(v_i\) is a dependent voltage source. \(v_e\) is an unknown sinusoidal voltage source. \(v_{u1}\) is an unmeasured exogenous voltage disturbance; \(v_{u2}\) is an unmeasured exogenous voltage disturbance in the output channel; \(i_s\) is a measured exogenous current disturbance. \(v_o\) is the voltage output. Our objective is to achieve the desired voltage output \(v_o\) by adjusting \(v_i\). Take \(\hat{x}_1 = i_1\), \(\hat{x}_2 = v_c\) as the state variables, \(u = v_i\) as the input, and \(y = v_o\) as the output, \(\tilde{w} = [v_{u1} \ v_{u2}]^T\) as the unmeasured exogenous disturbance, and \(\tilde{w} = i_s\) as the measured exogenous disturbance. The sinusoidal component \(v_e\) can be modeled by a second-order linear system: \(\ddot{x}_3 = x_4, \dot{x}_4 = \theta_f x_3\), where \(v_e = x_3\). The parameter \(\theta_f\) determines the frequency of the sinusoid, which is assumed to be unknown. For simulation purposes, we will set \(v_c(t) = \sqrt{2}\sin(2t + \pi/4)\), which corresponds to the choice of \(\dot{x}_3(0) = 1, \dot{x}_4(0) = 2, \text{ and } \theta_f = -4\). The initial conditions for \(\dot{x}_1\) and \(\dot{x}_2\) are set to be \(\dot{x}_1(0) = 1\) and \(\dot{x}_2(0) = 1\). Then, the circuit can be
modeled as
\[
\dot{x} = \begin{bmatrix}
-R/L & -1/L & 1/L & 0 \\
1/C & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \theta_f & 0 \\
\end{bmatrix} x + \begin{bmatrix}
1/L \\
0 \\
0 \\
0 \\
\end{bmatrix} u + \begin{bmatrix}
0 & 0 \\
0 & -1/C \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \dot{w}
\tag{50a}
\]
\[
y = [R \ 1 \ 0] \dot{x} + [1 \ 0] \dot{w}
\tag{50b}
\]

Define \(\theta = [\theta_1 \ \theta_2 \ \theta_3]'\), where \(\theta_1 = 1/C\), \(\theta_2 = \theta_f\), and \(\theta_3 = \theta_1 \theta_2 = \theta_f / C\). Then, the true values for the unknown parameter vector \(\theta\) is \([1 \ -4 \ -4]'\). We assume \(\theta_1\) belongs to the interval \([0.5 \ 6.5]\); \(\theta_2\) belongs to the interval \([-6 \ 0]\); and \(\theta_3\) belongs to the interval \([-12 \ 0]\). Then, the projection function \(P(\theta)\) is selected to be
\[
P(\theta) = \frac{1}{3} \left( \frac{(\theta_1 - 3.5)^2}{9} + \frac{(\theta_2 + 3)^2}{9} + \frac{(\theta_3 + 6)^2}{36} \right)
\]

Since the system is observable, the overall system (50) can be transformed into the observer canonical form through the transformation:
\[
x = \begin{bmatrix}
1 & 1 & 0 & 0 \\
\theta_1 & 0 & 1 & 0 \\
-\theta_2 & -\theta_2 & \theta_1 & 1 \\
-\theta_3 & 0 & 0 & \theta_1 \\
\end{bmatrix} \dot{x}, \quad w = \begin{bmatrix}
1/2.5 & 0 \\
\theta_1/30 & (\theta_1 - \theta_2)/30 \\
-\theta_2/15 & -\theta_2/15 \\
-\theta_3/30 & \theta_3/30 \\
0 & 1/2.5 \\
\end{bmatrix} \dot{w}
\]
Then, the observer canonical form of (50) is obtained as

\[
\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} 0 \\ 2.5 \\ 0 \\ 0 \end{bmatrix} w
\]

\[
y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 2.5 \end{bmatrix} w
\]

The reference trajectory is generated through the same reference model as that in the example of Section 7. We set the desired disturbance attenuation level \( \gamma = 12.5 > \zeta^{-1} = 2 \). The initial states for the controller are selected to be \( \dot{\theta}_0 = [1 \, 0 \, 0 \, 0]' \), \( \dot{\theta}_0 = [0 \, 0 \, 0 \, 0]' \), and \( Q_0 = 0.0005 I_3 \). For the adaptive control design, we select \( \rho_n = [0 \, 0 \, 0 \, 1]' \); \( K_e = 0.2344 \); \( \Delta_1 = I_4 \); \( \beta_0 = 0 \); \( \rho_o = 2 \); \( \gamma_1 = \beta_1 = 0.5 \); \( \varepsilon = 1 \); \( k_e = 0 \).

Since \( \varepsilon \) is 1, then \( \Sigma \) and \( s_\Sigma \) will be constant quantities. We have \( \Sigma = 12.5 I_3 \) and \( s_\Sigma = 0.2344 \). The matrices \( \Pi \) and \( A_f \) are calculated to be

\[
\Pi = \begin{bmatrix} 0.0496 & 0.1574 & 0.1691 & 0.0829 \\ 0.1574 & 1.1878 & 1.3746 & 0.7149 \\ 0.1691 & 1.3746 & 3.3729 & 2.0049 \\ 0.0829 & 0.7149 & 2.0049 & 2.1536 \end{bmatrix} \times 10^3, \quad A_f = \begin{bmatrix} -8.6205 & 1 & 0 & 0 \\ -24.1769 & 0 & 1 & 0 \\ -25.9704 & 0 & 0 & 1 \\ -12.7373 & 0 & 0 & 0 \end{bmatrix}
\]

\( k_e = 0 \) leads to \( \zeta_e = 0 \). The estimator dynamics are listed below:

\[
\dot{\eta} = A_f \eta + p_n y, \quad \eta(0) = 0_{4 \times 1}
\]

\[
\dot{\lambda} = A_f \lambda + p_n u, \quad \lambda(0) = 0_{4 \times 1}
\]

\[
\dot{\eta}_\hat{w} = A_f \eta_{\hat{w}} + p_n \hat{w}, \quad \eta_{\hat{w}}(0) = 0_{4 \times 1}
\]

\[
\Phi = [A_f^3 \eta \ldots A_f \eta \, \eta] M_f^{-1} \hat{A}_{2111} + [A_f^3 \lambda \ldots A_f \lambda \, \lambda] M_f^{-1} \hat{A}_{212}
\]

\[
\phi_{\eta} = [A_f^3 \eta \ldots A_f \eta \, \eta \eta] M_f^{-1} \hat{A}_{2131}
\]

\[
\dot{\theta} = -12.8(P_r (\tilde{\theta}) - 12.8 \Phi_1'(y_d - \tilde{x}_1)) + 320 \Phi_1'(y - \tilde{x}_1)
\]

\[
\dot{x}_1 = -12.8 \Phi_1 P_r (\tilde{\theta}) - \tilde{x}_1 + \tilde{x}_2 - (0.3175 + 12.8 \Phi_1 \Phi_1'(y_d - \tilde{x}_1) + u - \hat{w} \tilde{\theta}_1
\]

\[
+ (7.9381 + 320 \Phi_1 \Phi_1')(y - \tilde{x}_1)
\]
\[
\dot{x}_2 = -12.8\Phi_2 P_r(\ddot{\theta}) + \dot{x}_3 - (1.0074 + 12.8\Phi_2 \Phi'_1)(y_d - \ddot{x}_1) + y(-\ddot{\theta}_1 + \ddot{\theta}_2) + u\ddot{\theta}_1 \\
+ (25.1843 + 320\Phi_2 \Phi'_1)(y - \ddot{x}_1)
\]

\[
\dot{x}_3 = -12.8\Phi_3 P_r(\ddot{\theta}) + \dot{x}_4 - (1.0821 + 12.8\Phi_3 \Phi'_1)(y_d - \ddot{x}_1) + (y - u)\ddot{\theta}_2 + \dot{w}\ddot{\theta}_3 \\
+ (27.0524 + 320\Phi_3 \Phi'_1)(y - \ddot{x}_1)
\]

\[
\dot{x}_4 = -12.8\Phi_4 P_r(\ddot{\theta}) - (0.5307 + 12.8\Phi_4 \Phi'_1)(y_d - \ddot{x}_1) + (y - u)\ddot{\theta}_3 \\
+ (13.2681 + 320\Phi_4 \Phi'_1)(y - \ddot{x}_1)
\]

The reduced-order control law is given by

\[
\mu = -0.5\ddot{x}_1 - \ddot{x}_2 + (0.3175 + 12.8\Phi_1 \Phi'_1)(y_d - \ddot{x}_1) + 12.8\Phi_1 P_r(\ddot{\theta}) + \dot{w}\ddot{\theta}_1 + 1.5y_d + \dot{y}_d
\]

The simulation results are shown in Figure 6. First, we set the command input \( d(t) = 2.7\sin(t) \) and the disturbance \( \dot{w}(t) \) to be zero. Figures 6(a), (c), and (e) show the system responses achieved with reduced-order controller. Figures 6(b), (d), and (f) show the system responses achieved with full-order controller. We observe that the simulation results for both of the controllers are very similar. The tracking errors asymptotically converge to zero. That means the sinusoidal disturbance \( v_e \) is completely canceled. The parameter estimates also converge to their true values. The control inputs are bounded by 8. Both of the controllers achieve comparable transient and even comparable steady-state performance after long-term observation, which is different from the simulation results of the example in Section 7. To analyze the reason, we may conclude that, with increase of dynamic order of the system, the advantage of simplified controller structure is more prominent, which compensates its negative effects.

In case that the system is subject to arbitrarily varying disturbance \( \dot{w} \), we observe that both types of controllers have comparable system responses as well. Their transient responses behave well. Their parameter estimates all oscillate around their true values in steady state.

9. CONCLUSIONS

In this paper, we have studied the reduced-order adaptive control design for SISO linear systems with noisy output measurements. The main contribution of this paper is that the proposed order-reduction method can simplify the controller structure by \( n \) integrators without any additional assumptions or sacrifice in the strong robustness properties of the closed-loop system, in comparison with the full-order controller designed in [32, 35]. This control problem is formulated into the framework of nonlinear \( H^\infty \)-optimal control under imperfect state measurements, where the unknown parameter vector is viewed as part of the expanded state vector, and the objectives of robust adaptive control are then incorporated into the optimization of a single soft-constrained game-theoretic cost function. The adaptive control design is carried out in two steps: first step is the estimation design step and second step is the controller design step. The estimation design step is almost essentially the same as that of the full-order adaptive control design. The main difference between the reduced-order control design discussed in this paper and the full-order control design lies in the controller design step. At this step, although we employ the backstepping
Figure 6. System response of circuit example under $d(t) = 2.7\sin(t)$ and no arbitrarily varying disturbance $\dot{w}$ for reduced-order controller (left-column plots) and full-order controller (right-column plots): (a), (b) tracking error; (c), (d) parameter estimate; solid line for $\hat{\theta}_1$; dash line for $\hat{\theta}_2$; dash dot line for $\hat{\theta}_3$; and (e), (f) control input.
methodology as the full-order controller design, we start the controller design from step 1, without first stabilizing \( \eta \) dynamics as step 0 does in full-order controller design. Then, there is no need to generate \( \eta_d \) dynamics for \( \eta \) to track; the dynamic order of the controller is thereby reduced by \( n \). Step 0 of the full-order controller design may be skipped that does not affect the robustness of the closed-loop system because the dynamics of \( \ddot{x} - \Phi \dot{\theta} \) admits desired structure which may be substituted for the \( \hat{\eta} \) dynamics. On the other hand, the dynamics of \( \ddot{x} - \Phi \dot{\theta} \) have an undesirable feature that they depend also on \( \hat{\xi} \). Then, the trade-off for the order reduction is that \( \hat{\xi} \) may not be set to the optimal choice. Also, the lack of step 0 leads to the disappearance of a nonpositive drift term related to \( \eta \) dynamics in the derivative of the closed-loop value function. All of these trade-offs may be responsible for the degradation of the closed-loop responses after order reduction, which can be observed in the examples presented in this paper. However, as mentioned above, the reduced-order controller is shown to preserve exactly the same strong robustness properties as those of the full-order controller. These properties encompass the following three aspects: boundedness of all closed-loop signals, desired disturbance attenuation property, and asymptotic tracking property. Two examples are presented to further corroborate the theoretical findings. The order-reduction scheme proposed in this paper may be further applied to SISO linear systems with repeated noisy measurements [36] and a special class of MIMO linear systems [37].

APPENDIX A

A.1. Proof of Lemma 1 (for recursive backstepping design)

Proof

\( \forall (x_o, x_d) \in D_1, \forall x_a \in D_a, \forall w \in D_w \), we have

\[
\dot{V}_o(x_o, x_a, x_d, w) = \frac{\partial V_o}{\partial x_o}(x_o)(f_o(x_o, x_a, x_d) + h_o(x_o, x_a, x_d)w)
\]

\[
= \frac{\partial V_o}{\partial x_o}(x_o)(f_o(x_o, x_a, x_d) + h_o(x_o, x_a, x_d)w) + \frac{\partial V_o}{\partial x_o}(x_o)(\tilde{f}_o(x_o, x_a, x_d)
\]

\[
+ \tilde{h}_o(x_o, x_a, x_d)w)(x_a - \zeta_o(x_a))
\]

where \( \tilde{f}_o \) and \( \tilde{h}_o \) are \( \mathcal{C}^k \) functions (see [8, p. 433]) on \( D_o \times D_a \times D_d \) and defined by

\[
\tilde{f}_o(x_o, x_a, x_d) := \begin{cases} 
\frac{f_o(x_o, x_a, x_d) - f_o(x_o, \zeta_o(x_o), x_d)}{x_a - \zeta_o(x_o)}, & x_a \neq \zeta_o(x_o) \\
\frac{\partial f_o}{\partial x_a}(x_o, \zeta_o(x_o), x_d), & x_a = \zeta_o(x_o)
\end{cases}
\]

\[
\tilde{h}_o(x_o, x_a, x_d) := \begin{cases} 
\frac{h_o(x_o, x_a, x_d) - h_o(x_o, \zeta_o(x_o), x_d)}{x_a - \zeta_o(x_o)}, & x_a \neq \zeta_o(x_o) \\
\frac{\partial h_o}{\partial x_a}(x_o, \zeta_o(x_o), x_d), & x_a = \zeta_o(x_o)
\end{cases}
\]
By the assumption, we have
\[
\dot{V}(x_0, x_a, x_d, u, w) = -l_o(x_o, x_d) + \gamma^2 |w|^2 - \gamma^2 |w - \sigma_o(x_o, x_d)|^2 + \frac{\partial V_o}{\partial x_o}(x_o) (\hat{f}(x_o, x_a, x_d)
\]
\[
\times \tilde{h}_o(x_o, x_a, x_d) w)(x_o - z_o(x_o))
\]

Let \( z = x_d - z_o(x_o) \). Then, \( \forall u \in D_o \),
\[
\dot{V}(x_0, x_a, x_d, u, w) = -l_o(x_o, x_d) + \gamma^2 |w|^2 - \gamma^2 |w - \sigma_o(x_o, x_d)|^2 + z(\chi_1 + 2\gamma^2 \chi_2 w + \chi_3 u)
\]
where \( \chi_1, \chi_2, \) and \( \chi_3 \) are \( \mathcal{C}^k \) functions on \( D_o \times D_a \times D_d \) and given by
\[
\chi_1(x_0, x_a, x_d) = 2\delta(x_o, x_a) \left( \frac{\partial \delta}{\partial x_o}(x_o, x_a) - \delta(x_o, x_a) \frac{\partial \sigma_o}{\partial x_o}(x_o) \right) f_o(x_o, x_a, x_d)
\]
\[
+ 2\delta(x_o, x_a) \pi_\delta(x_o, x_a) f_o(x_o, x_a, x_d) + \frac{\partial V_o}{\partial x_o}(x_o) \tilde{f}_o(x_o, x_a, x_d) \quad (A1a)
\]
\[
\chi_2(x_0, x_a, x_d) = \gamma^{-2} \delta(x_o, x_a) \left( \frac{\partial \delta}{\partial x_o}(x_o, x_a) - \delta(x_o, x_a) \frac{\partial \sigma_o}{\partial x_o}(x_o) \right) \tilde{h}_o(x_o, x_a, x_d)
\]
\[
+ \gamma^{-2} \delta(x_o, x_a) \pi_\delta(x_o, x_a) \tilde{h}_o(x_o, x_a, x_d)
\]
\[
+ 2\gamma^{-2} \delta(x_o, x_a) \frac{\partial V_o}{\partial x_o}(x_o) \tilde{h}_o(x_o, x_a, x_d) \quad (A1c)
\]
\[
\chi_3(x_0, x_a, x_d) = 2\delta(x_o, x_a) \pi_\delta(x_o, x_a) \tilde{g}_o(x_o, x_a, x_d)
\]
\[
(A1d)
\]
Define \( z: D_o \times D_a \times D_d \to \mathbb{R} \) by
\[
z(x_0, x_a, x_d) = (\chi_3(x_0, x_a, x_d) )^{-1} ( -\chi_1(x_0, x_a, x_d) - 2\gamma^2 \chi_2(x_0, x_a, x_d) \sigma_o(x_o, x_d)
\]
\[
- \gamma^2 \chi_2(x_0, x_a, x_d) (\chi_2(x_0, x_a, x_d))' z - \phi(x_0, x_a, x_d)) \quad (A2)
\]
Then, it implies that
\[
\dot{V}(x_0, x_a, x_d, u, w)|_{u=z(x_0, x_a, x_d)} = -l_o(x_o, x_d) - \phi(x_o, x_a, x_d) z + \gamma^2 |w|^2 - \gamma^2 |w - \sigma_o(x_o, x_d)|^2 - (\chi_2(x_0, x_a, x_d))' z|^2
\]
Equation (A2) holds by defining \( \sigma(x_o, x_a, x_d) = \sigma_o(x_o, x_d) + (\chi_2(x_0, x_a, x_d))' (x_o - z_o(x_o)) \), which leads to Equation (28), and is \( \mathcal{C}^k \) on \( D_o \times D_a \times D_d \).

If, in addition, there exists \( (x_o0, x_a0, x_d0) \in D_o \times D_a \times D_d \), such that \( (\partial V_o/\partial x_o)(x_o0) = 0 \times \mathbb{R}_1 \), \( f_o(x_o0, x_a0, x_d0) = 0 \times \mathbb{R}_1 \), \( f_o(x_o0, x_a0, x_d0) = 0 \), \( z_o(x_o0) = x_o0 \), and \( \phi(x_o0, x_a0, x_d0) = 0 \), then \( \chi_1(x_o0, x_a0, x_d0) = 0 \), \( \sigma_o(x_o0, x_d0) = 0 \times \mathbb{R}_1 \), and \( x_o0 - z(x_o0) = 0 \). This further implies that \( z(x_o0, x_a0, x_d0) = 0 \).

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A2. Useful lemmas

Lemma 2
Consider the following linear time-invariant system:

\[ \dot{z} = Az + Bu + Dw \]  \hspace{1cm} (A3a)
\[ y = Cz + Ew \]  \hspace{1cm} (A3b)

where \( z \) is an \( n \)-dimensional state vector, \( n \in \mathbb{N} \); \( u \) and \( y \) are scalar input and output, respectively. \( w \) is a \( q \)-dimensional unknown disturbance, \( q \in \mathbb{N} \); the matrix \( A \) is Hurwitz. The transfer function \( H(s) = C(sI_n - A)^{-1}B \) is SMP and has relative degree \( r \in \mathbb{N} \). Then, there exists a real invertible coordinate transformation matrix \( T \) such that system (A3) can be transformed into the form (A10), and which satisfies Assumptions 6 with index \( r_1 = r \).

Proof
First, we partition the system dynamics (A3) into observable and unobservable parts through the state transformation

\[ \begin{bmatrix} x'_o \\ x'_\bar{o} \end{bmatrix} = T o \begin{bmatrix} x \end{bmatrix} \]

with a real invertible matrix \( T_o \). Then system (A3) admits representation

\[ \begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{o\bar{o}} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u + \begin{bmatrix} D_o \\ D_{\bar{o}} \end{bmatrix} w \]  \hspace{1cm} (A4a)
\[ y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Ew \]  \hspace{1cm} (A4b)

where the state vector \( x_o \) is \( n_o \)-dimensional, \( n_o \in \mathbb{N} \), and the partitioning of the matrices is compatible with the partitioning of \( [x'_o \ x'_\bar{o}]' \). The pair \( (A_o, C_o) \) is observable.

Now we only consider the observable part in (A4). There exists a real invertible coordinate transformation \( \begin{bmatrix} x'_{co} \\ x'_{\bar{co}} \end{bmatrix} = T_{co}^{-1}x_o \) such that the observable part is further divided into the controllable and uncontrollable parts as follows:

\[ \begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{\bar{co}} \end{bmatrix} = \begin{bmatrix} A_{co} & A_{c\bar{co}} \\ 0 & A_{\bar{co}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{\bar{co}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ 0 \end{bmatrix} u + \begin{bmatrix} D_{co} \\ D_{\bar{co}} \end{bmatrix} w \]  \hspace{1cm} (A5a)
\[ y = \begin{bmatrix} C_{co} & C_{c\bar{co}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{\bar{co}} \end{bmatrix} + Ew \]  \hspace{1cm} (A5b)

where the state vector \( x_{co} \) is \( n_c \)-dimensional, \( n_c \in \mathbb{N} \), and the partitioning of the matrices is compatible with that of \( [x'_{co}, x'_{\bar{co}}]' \); \( A_{\bar{co}} \) is Hurwitz. The triple \( (A_{co}, B_{co}, C_{co}) \) is observable and controllable. Clearly, the transfer function \( H(s) = C_{co}(sI_{n_c} - A_{co})^{-1}B_{co} \). Let the transfer function be given by

\[ H(s) = C_{co}(sI_{n_c} - A_{co})^{-1}B_{co} = \frac{b_0 s^{n_c-r} + b_1 s^{n_c-r-1} + \cdots + b_{n_c-r}}{s^{n_c} + d_1 s^{n_c-1} + \cdots + a_{n_c}}, \quad b_0 \neq 0 \]  \hspace{1cm} (A6)
By Lemma 12 in [32], there exists a real invertible matrix $T_1$ such that

$$
\begin{bmatrix}
T_1^{-1} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A_{co} & B_{co} \\
C_{co} & 0
\end{bmatrix}
\begin{bmatrix}
T_1 & 0 \\
0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\
\hat{A}_{21} & \hat{A}_{22} & 0 \\
\hat{C}_1 & 0 & 0
\end{bmatrix}
$$

(A7)

where

$$
\hat{A}_{11} = \begin{bmatrix}
a_1 & \vdots & I_{r-1} \\
a_r & 0_{1 \times (r-1)}
\end{bmatrix},
\hat{A}_{12} = \begin{bmatrix}
0 \\
1
\end{bmatrix},
\hat{A}_{21} = \begin{bmatrix}
a_{r+1} & \vdots & 0_{(n_c-r) \times (r-1)} \\
a_n & \vdots & 0_{1 \times (r-1)}
\end{bmatrix},
\hat{A}_{22} = \begin{bmatrix}
-b_1/b_0 & \vdots & I_{n_c-r-1} \\
-b_{n_c-r-1}/b_0 & \vdots & 0_{1 \times (n_c-r-1)}
\end{bmatrix}
$$

$\hat{B}_1 = [0_{1 \times (r-1)} \ b_0]^\top$, $\hat{C}_1 = [1 \ 0_{1 \times (r-1)}]$

where $\hat{A}_{22}$ is Hurwitz, because the transfer function $H(s)$ is SMP.

Then the desired transformation matrix $T$ is equal to

$$
T = T_o
\begin{bmatrix}
T_c & 0 \\
0 & I_{n_o-n_c}
\end{bmatrix}
\begin{bmatrix}
T_1 & 0 \\
0 & I_{n_o-n_c} \\
0 & 0 \\
0 & I_{n_o-n_c}
\end{bmatrix}
$$

(A8)

Then via the transform matrix $[x'_1 \ x'_2 \ x'_{co} \ x'_{o}]^\top = Tz$, the system can be transformed into

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_{co} \\
\dot{x}_o
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & 0 \\
\hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & 0 \\
0 & A_{co} & 0 & 0 \\
\hat{A}_{41} & \hat{A}_{42} & \hat{A}_{43} & A_o
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_{co} \\
x_o
\end{bmatrix} +
\begin{bmatrix}
\hat{B}_1 \\
0 \\
0 \\
0
\end{bmatrix} u +
\begin{bmatrix}
\hat{D}_1 \\
\hat{D}_2 \\
D_{co} \\
D_o
\end{bmatrix} w
$$

(A9a)

$$
y = [\hat{C}_1 \ 0 \ C_{co} \ 0] \begin{bmatrix}
x_1 \\
x_2 \\
x_{co} \\
x_o
\end{bmatrix} + E w
$$

(A9b)
where all matrices are constant matrices of appropriate dimension. Clearly, (60) admits the state space representation (A10) and satisfies Assumption 6 with index \( r_1 = r \). □

The lemmas presented below are first derived from [32]. We consider two linear time-invariant systems with same inputs. The first linear system admits the state space representation:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5
\end{bmatrix} =
\begin{bmatrix}
A_{111} & A_{112} & 0 \\
A_{121} & A_{122} & 0 \\
0_{n \times n} & A_{133} & A_{134} \\
0 & 0 & A_{144}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} +
\begin{bmatrix}
B_{11} \\
B_{12} \\
0 \\
0 \\
B_{15}
\end{bmatrix}
\begin{bmatrix}
u \n\hat{w}
\end{bmatrix} +
\begin{bmatrix}
D_{11} \\
D_{12} \\
0 \\
0 \\
D_{15}
\end{bmatrix}
\begin{bmatrix}
\hat{w}
\end{bmatrix}
\]

\( \bar{x}_1(0) = \bar{x}_{10}, \ \bar{x}_2(0) = \bar{x}_{20}, \ \bar{x}_3(0) = \bar{x}_{30}, \ \bar{x}_4(0) = \bar{x}_{40}, \ \bar{x}_5(0) = \bar{x}_{50} \) \hspace{0.5cm} (A10a)

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} =
\begin{bmatrix}
C_{11} & 0 & C_{13} & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x}_1' \\
\bar{x}_2' \\
\bar{x}_3' \\
\bar{x}_4'
\end{bmatrix} + E_1 \hat{w} \hspace{0.5cm} (A10b)
\]

where \([\bar{x}_1' \ldots \bar{x}_4']\) is the \( n_1 \)-dimensional state vector; \( \bar{x}_1 \) is \( r_1 \)-dimensional; \( \bar{x}_2 \) is \((n_{11} - r_1)\)-dimensional; \([\bar{x}_3' \bar{x}_4']\) is \( n_{12} \)-dimensional; \( \bar{x}_3 \) is \( n_{13} \)-dimensional; \( \bar{u} \) is a scalar input; \( \hat{w} \) is a \( q_1 \)-dimensional unknown disturbance input; \( \bar{y} \) is a scalar output; all signals are continuous on a nonempty and possibly infinite interval \([0, t_f]\); all partitions of the matrices are compatible with the partition of the state vector; \( n_1, n_{11}, r_1, q_1 \in \mathbb{N} \); and \( n_{11} - r_1, n_{12}, n_{13} \in \mathbb{N} \cup \{0\} \). Denote the elements of \( \bar{x}_1 \) by \([\bar{x}_{11} \ldots \bar{x}_{1r_1}]\). The following assumption is imposed on (A10).

**Assumption 6**
The matrices \( A_{111}, A_{121}, B_{11}, \) and \( C_{11} \) admits the following structure.

\[
A_{111} =
\begin{bmatrix}
\begin{array}{c|c}
I_{r_1 - 1} & 0_{1 \times (r_1 - 1)} \\
0_{1 \times (r_1 - 1)} & 0_{(n_{11} - r_1) \times (r_1 - 1)}
\end{array}
\end{bmatrix}
\hspace{0.5cm} A_{121} =
\begin{bmatrix}
0_{(n_{11} - r_1) \times (r_1 - 1)} & 0_{r_1 \times (n_{11} - r_1)} \\
0_{r_1 \times (n_{11} - r_1)} & 0_{r_1 \times r_1}
\end{bmatrix}
\hspace{0.5cm} (A11a)
\]

\[
B_{11} =
\begin{bmatrix}
0_{(r_1 - 1) \times 1} \\
b_0
\end{bmatrix}
\hspace{0.5cm} C_{11} =
\begin{bmatrix}
0_{1 \times (r_1 - 1)} \\
1
\end{bmatrix}
\hspace{0.5cm} (A11b)
\]

where \( b_0 \neq 0 \). Matrices \( A_{122} \) and \( A_{133} \) are Hurwitz (empty matrix is considered to be Hurwitz). Matrix \( A_{144} \) has all of its eigenvalues lying on the imaginary axis, each is associated with Jordan block(s) of order 1.

The second linear system admits a state space representation:

\[
\begin{align*}
\dot{\bar{\eta}} &= A_2 \bar{\eta} + B_2 \bar{u} + D_2 \hat{w}, \\
\bar{\eta}(0) &= \bar{\eta}_0 \hspace{0.5cm} (A12a)
\end{align*}
\]

\[
\begin{align*}
\ddot{\bar{z}} &= C_2 \bar{\eta} + E_2 \hat{w} \hspace{0.5cm} (A12b)
\end{align*}
\]
where $\bar{\eta}$ is the $n_2$-dimensional state vector with $n_2 \in \mathbb{N}$; $\xi$ is a scalar output; all signals are continuous on $[0, t_f)$; and the inputs $\bar{u}$ and $\bar{w}$ are the same as those of system (A10).

**Assumption 7**
The matrix $A_2$ is Hurwitz. The transfer function from $\bar{u}$ to $\bar{z}$, $C_2(sI_{n_2} - A_2)^{-1}B_2$ has relative degree $r_2 \in \mathbb{N}$.

According to the above assumptions, the following bounding lemma (derived in [32]) is valid.

**Lemma 3**
Consider the two real linear time-invariant systems (A10) and (A12). Let Assumptions 6 and 7 hold for systems (A10) and (A12), respectively. Consider the time interval $[0, t_f)$, where $t_f \in \mathbb{R}$. Then, the following results hold.

(i) If there exists a constant $c \geq 0$, such that $|\bar{w}(t)| \leq c$, $|\bar{y}(t)| \leq c, \forall t \in [0, t_f)$, $|\bar{x}_{10}| \leq c$, $|\bar{x}_{20}| \leq c$, $|\bar{x}_{30}| \leq c$, $|\bar{x}_{40}| \leq c$, $|\bar{h}_0| \leq c$, and $r_1 \leq r_2$, then, there exists a constant $\overline{c}_1 \geq 0$ such that $|\bar{\zeta}(t)| \leq \overline{c}_1$, $\forall t \in [0, t_f)$. (The norm of an empty matrix is set to 0.)

(ii) If there exists a constant $c \geq 0$, such that $|\bar{w}(t)| \leq c$, $|\bar{x}_{1k}(t)| \leq c, k = 1, \ldots, k_0, \forall t \in [0, t_f)$, with $1 \leq k_0 \leq r_1$, $|\bar{x}_{10}| \leq c$, $|\bar{x}_{20}| \leq c$, $|\bar{x}_{30}| \leq c$, $|\bar{x}_{40}| \leq c$, $|\bar{h}_0| \leq c$, and $r_1 - k_0 + 1 \leq r_2$, then, there exists a constant $\overline{c}_2 \geq 0$ such that $|\bar{\zeta}(t)| \leq \overline{c}_2$, $\forall t \in [0, t_f)$.

The linear system (A10) is called the reference system in the application of this lemma.

**Lemma 4**
Consider the linear system (1) under Assumption 1. Let its output be the input of the following real linear time-invariant system:

$$\dot{\tilde{\lambda}} = A_3 \tilde{\lambda} + B_3 y + D_3 \dot{w}$$

$$\tilde{\pi} = C_3 \tilde{\lambda} + K_3 y + E_3 \dot{w}$$

where $\tilde{\lambda}$ is an $n_3$-dimensional state vector; $n_3 \in \mathbb{N} \cup \{0\}$; $\tilde{\pi}$ is a scalar output; $\dot{w}$ is the same disturbance input as in (1); matrix $A_3$ is Hurwitz; and the transfer function from $y$ to $\tilde{\pi}$, $H(s) = C_3(sI_{n_3} - A_3)^{-1}B_3 + K_3$, is SMP with relative degree $r_3 \in \mathbb{N} \cup \{0\}$. Then the composite system with state vector $[\tilde{\lambda}' \ y']'$, input $u$ and $\dot{w}$, and output $\tilde{\pi}$, admits a state space representation (A10) satisfying Assumption 6, under some real coordinate transformation, such that $r_1 = r + r_3$; and the matrix $A_{155}$ is Hurwitz.

**Lemma 5**
Let $n \in \mathbb{N}$, $D \subseteq \mathbb{R}^n$ be nonempty, $[t_0, t_1) \subset \mathbb{R}$ be a nonempty interval, and $K \subseteq D$ be compact. Let $\zeta : [t_0, t_1) \rightarrow D$ be continuous, $V : [t_0, t_1) \times D \rightarrow \mathbb{R}$ be nonnegative and continuous, and $W_i : D \rightarrow \mathbb{R}$ be nonnegative and continuous, $i = 1, 2$. Assume that

(i) $W_1(x) \leq V(t, x) \leq W_2(x), \forall (t, x) \in [t_0, t_1) \times D$;

(ii) $S_{12} := \{x \in D : W_1(x) \leq x\}$ is compact, $\forall x \in \mathbb{R}$;

(iii) $\forall \epsilon \in [t_0, t_1)$, with $\bar{\zeta}(t) \in D \setminus K$ implies $\limsup_{h \rightarrow 0^+} (V(t + h, \bar{\zeta}(t + h)) - V(t, \bar{\zeta}(t))) / h < 0$.

Then, there exists a constant $\eta \geq 0$, such that $V(t, \bar{\zeta}(t)) \leq \eta, \forall t \in [t_0, t_1)$. Furthermore, $\bar{\zeta}(t) \in S_{1\eta}$, $\forall t \in [t_0, t_1)$ and $S_{1\eta} \subseteq D$ is compact.
REFERENCES


