Adaptive controller design and disturbance attenuation for SISO linear systems with zero relative degree under noisy output measurements

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SUMMARY

In this paper, we present robust adaptive controller design for SISO linear systems with zero relative degree under noisy output measurements. We formulate the robust adaptive control problem as a nonlinear $H^\infty$-optimal control problem under imperfect state measurements, and then solve it using game theory. By using the a priori knowledge of the parameter vector, we apply a soft projection algorithm, which guarantees the robustness property of the closed-loop system without any persistency of excitation assumption of the reference signal. Owing to our formulation in state space, we allow the true system to be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and the uncontrollable modes on the $j\omega$-axis are uncontrollable from the exogenous disturbance input. This assumption allows the adaptive controller to asymptotically cancel out, at the output, the effect of exogenous sinusoidal disturbance inputs with unknown magnitude, phase, and frequency. These strong robustness properties are illustrated by a numerical example. Copyright © 2009 John Wiley & Sons, Ltd.

1. INTRODUCTION

The design of adaptive controllers has been an important research topic since 1970s. The classic adaptive control design based on the certainty equivalence principle [1] has been proven to be successful especially for the linear systems with or without stochastic disturbance inputs [2–7]. This approach leads to structurally simple adaptive controllers. Yet, early designs based on this approach has been shown to be nonrobust [8, 9] when the system is subject to exogenous disturbance inputs and unmodeled dynamics. Then, the stability and the performance of a system under disturbance and/or uncertainty becomes an important issue. This motivates the study of robust adaptive control, which has attracted significant research attention since 1980s. Also, this approach fails to generalize to systems with severe nonlinearities. This motivates the study of nonlinear adaptive control in 1990s.

Robust adaptive control has been an important research topic in the late 1980s and the early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust [10–12]. Despite their
successes, they fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The topic of nonlinear adaptive control has been widely studied in the last decade after the celebrated characterization of feedback linearizable or partially feedback linearizable systems [13]. The introduction of the integrator backstepping methodology [14] allows us to design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems systematically. Since then, a lot of important contributions were motivated by this approach, and a complete list of references can be found in the book [15]. Moreover, this approach has been applied to linear systems to compare performance with the certainty equivalence approach. However, this approach has also been shown to be nonrobust [16] when the system is subject to exogenous disturbance inputs.

$H_\infty$-optimal control has been proposed as a solution to the robust control problem. The game-theoretic approach to $H_\infty$-optimal control [17] developed for the linear quadratic problems offers the most promising tool to generalize the results to nonlinear systems [18–20]. The worst-case analysis approach to adaptive control was proposed to address the disturbance attenuation properties of the closed-loop system directly. In this approach, the robust adaptive control problem is formulated as a nonlinear $H_\infty$-optimal control problem under imperfect state measurements. Using cost-to-come function analysis, it can be converted into a problem under full information measurements. This full information measurement problem is then solved for a suboptimal solution using the integrator backstepping methodology. This design paradigm has been applied to worst-case parameter identification problems [21], which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems [22–26], which has led to new classes of parametrized robust adaptive controllers for linear and nonlinear systems.

In this paper, we study the adaptive control design for SISO linear systems with zero relative degree under noisy output measurements using a similar approach as that of [23]. We follow the adaptive control design paradigm discussed in the above paragraph, and the closed-loop adaptive system possesses the following strong robustness properties. The closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where the ultimate attenuation lower bound for the achievable performance level is equal to the noise intensity in the measurement channel. All closed-loop signals are bounded for bounded disturbance input and bounded reference trajectory. Furthermore, it achieves asymptotic tracking of uniformly continuous and bounded reference trajectories for all bounded disturbance inputs that are of finite energy. This result has significant impact on active noise cancelation problems. That is, when the true system is subject to disturbances generated by an unknown exogenous linear system, we can extend our system model to include the states of the exogenous system as part of the model, and then asymptotically cancel out the effect of the noise at the output. This feature is illustrated by an example in the paper. Another contribution of this paper is that we address the disturbance attenuation property of the closed-loop system with corrupted measurement directly, and introduce the disturbance attenuation level as the noise cancelation performance index for SISO linear systems with zero relative degree. Moreover, it is envisioned that the proposed controller can be generalized in [26] to design a robust adaptive controller for SISO linear system subjected to plant and controller uncertainties.

The balance of the paper is organized as follows. In Section 2, we list the notations to be used in this paper. In Section 3, we formulate adaptive control problem and discuss the general solution methodology. In Section 4, we present the estimation and control design using cost-to-come function methodology. In Section 5, we present the main result of the paper, which states the robustness properties of the closed-loop system. The theoretical results are illustrated by one numerical example in Section 6. The paper ends with some concluding remarks in Section 7, and one Appendix.

2. NOTATIONS

We denote $\mathbb{R}$ to be the real line, $\mathbb{N}$ to be the set of natural numbers, $\mathbb{C}$ to be the set of complex numbers. For a function $f$, we say that it belongs to $\mathbb{C}$ if it is
For any symmetric matrix $M$, $\rightarrow M$ denotes its transpose. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{N}$, $|z|$ denotes $(z'z)^{1/2}$. For any vector $z \in \mathbb{R}^n$, and any $n \times n$-dimensional symmetric matrix $M$, where $n \in \mathbb{N}$, $|z|_M^2 = z'Mz$. For any matrix $M$, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix $M$, $\overrightarrow{M}$ denotes the vector formed by stacking up the column vector of the lower triangular part of $M$. For $n \times n$-dimensional symmetric matrices $M_1$ and $M_2$, where $n \in \mathbb{N}$, we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive-definite matrices is denoted by $\mathcal{S}_+$. For $n \in \mathbb{N} \cup \{0\}$, $I_n$ denotes the $n \times n$-dimensional identity matrix. For any matrix $M$, $||M||_p$ denotes its $p$-induced norm, $1 \leq p \leq \infty$. $\mathcal{L}_2^n$ denotes the set of square integrable functions and $\mathcal{L}_\infty^n$ denotes the set of bounded functions. For any $n, m \in \mathbb{N}$, $0_{n \times m}$ denotes the $n \times m$-dimensional matrix whose elements are zeros.

3. PROBLEM FORMULATION

We consider the adaptive control problem for single-input and single-output (SISO) linear time-invariant systems. We make the following assumption on the unknown system.

Assumption 1
The linear system is known to be at most $n$ dimensional, $n \in \mathbb{N}$.

We consider the following true system dynamics:

\begin{align*}
\dot{x} &= \hat{A}x + \hat{B}u + \hat{D}w, \quad x(0) = \bar{x}_0 \quad (1a) \\
y &= \hat{C}x + b_0u + \hat{E}w \quad (1b)
\end{align*}

where $\hat{x}$ is the $\bar{n}$-dimensional state vector, $\bar{n} \in \mathbb{N} \cup \{0\}$; $u$ is the scalar control input, $b_0 \in \mathbb{R}$ and $b_0 \neq 0$, $y$ is the scalar measurement output $\hat{w}$ is the $\hat{q}$-dimensional unmeasured disturbance input vector, $\hat{q} \in \mathbb{N}$, the state $\hat{x}$ has initial condition $\bar{x}_0$, and all input and output signals $y$, $u$, and $\hat{w}$ are continuous, the matrices $A, B, C, D$, and $\hat{E}$ are of the appropriate dimensions, generally unknown or partially unknown. The transfer function from $u$ to $y$ is $H(s) = \hat{C}(sI_{\bar{n}} - \hat{A})^{-1} \hat{B} + b_0$; $\bar{n} \in \mathbb{N}$ or $H(s) = b_0$; $\bar{n} = 0$.

Assumption 2
\(^1\)The pair $(\hat{A}, \hat{C})$ is observable. The transfer function $H(s)$ is known to have relative degree 0, and is strictly minimum phase. Moreover, the uncontrollable part (with respect to $u$) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to the eigenvalues of the matrix $\hat{A}$ on the $j \omega$-axis are uncontrollable from $\hat{w}$.

Without loss of generality, we will assume that the true system (1a) is of order $n$ based on Lemma A.2.

Since $(\hat{A}, \hat{C})$ is observable, there always exist a state diffeomorphism $x = T^{-1}{\hat{x}}$ and a disturbance transformation $w = M\hat{w}$, where $T$ is an unknown real invertible matrix and $M$ is a real $q \times q$-dimensional unknown matrix with $q \in \mathbb{N}$, such that system (1a) can be transformed into the following form in the $x$ coordinate with inputs $u$ and $w$:

\begin{align*}
\dot{x} &= Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta \\
&\quad + Bu + Dw, \quad x(0) = x_0 \quad (2a) \\
y &= Cx + u\bar{C}_1\theta + b_\rho u + Eu \quad (2b)
\end{align*}

where $\theta$ is the $\sigma$-dimensional vector of unknown parameters of the system, $\sigma \in \mathbb{N}$; the matrices $A, \bar{A}_{211}, \bar{A}_{212}, B, D, C, E, \text{ and } \bar{C}_1$ are of appropriate dimensions and completely known, and $b_\rho \in \mathbb{R}$ is also known. In addition, the high-frequency gain of the transfer function $H(s), b_0$, is equal to $b_\rho + \bar{C}_1\theta$. We will design the adaptive controller based on system (2a), which is called the design model. By Assumption 2, the pair $(A, C)$ is observable.

We have the following assumptions about the design model and reference signal $y_d$.

Assumption 3
$EE' > 0$. Define $\zeta := (EE')^{-1/2}$ and $L := DE'$.

\(^1\)When $\bar{n} = 0$, Assumption 2 is considered satisfied.
Assumption 4
The sign of the high-frequency gain $b_0$ is known. There exists a known smooth nonnegative radially unbounded strictly convex function $P: \mathbb{R}^n \to \mathbb{R}$, such that the true value of $\theta$ belongs to the set $\Theta := \{ \theta \in \mathbb{R}^n | P(\theta) \leq 1 \}$. Furthermore, for any $\tilde{\theta} \in \Theta$, we have $\text{sgn}(b_0)(b_{p0} + \tilde{C}_1 \tilde{\theta}) > 0$.

Assumption 5
The reference trajectory, $y_d$, is continuous, and available for feedback.

For system (1a), under Assumptions 1–5, the control law is generated by

$$u(t) = \mu(y[t_0,t], y_d[t_0,t])$$  \hspace{1cm} (3)

Furthermore, it must satisfy the following condition. For any uncertainty $(x_0, \theta, \dot{w}[0,\infty), y_d[t_0,\infty]) \in \mathcal{W} := \mathbb{R}^n \times \Theta \times \mathcal{G} \times \mathcal{G}$, which comprises the initial state, the true value of the unknown parameter vector, the unknown disturbance input waveform, and the reference trajectory, there must be a unique solution $\dot{x}_0[0,\infty]$ for the closed-loop system, which results in a continuous control input waveform $u[t_0,\infty)$. We denote the class of these admissible controllers by $\mathcal{M}_u$.

The objectives of our control design are to make the output of the system, $Cx + b_0u$, to asymptotically track the reference trajectory $y_d$, and guarantee the boundedness of all closed-loop signals, while rejecting the uncertainty $(x_0, \theta, \dot{w}[0,\infty), y_d[t_0,\infty]) \in \mathcal{W}$. For design purposes, instead of attenuating the effect of $\dot{w}$, we design the adaptive controller to attenuate the effect of $w$. We take the uncertainty $(x_0, \theta, \dot{w}[0,\infty), y_d[t_0,\infty])$ to belong to the set $\mathcal{W} := \mathbb{R} \times \Theta \times \mathcal{G} \times \mathcal{G}$. All of these objectives can be captured by the optimization of a single game-theoretic cost function, defined as follows.

**Definition 1**
A controller $\mu \in \mathcal{M}_u$ is said to achieve disturbance attenuation level $\gamma$ if there exists a nonnegative function $l(t, \theta, x, y[t_0,t], y_d[t_0,t])$ such that

$$\sup_{(x_0, \theta, \dot{w}[0,\infty), y_d[t_0,\infty]) \in \mathcal{W}'} J_{\gamma t} \leq 0, \ \forall t \geq 0$$  \hspace{1cm} (4)

and $l(t, \theta, x, y[t_0,t], y_d[t_0,t]) \geq 0$ along the closed-loop trajectory, where

$$J_{\gamma t} := \int_0^{t_f} ((C x(t) + u(t) \tilde{C}_1 \theta + b_{p0} u(t) - y_d(t))^2 + I(t, \theta, x(t), y[t_0,t], y_d[t_0,t]) - y_d(t))^2 d\tau$$

Furthermore, $y'[t_0,t] = \tilde{C}(u) x(t) + b_{p0} u(t) + E w$

where $\tilde{\theta}_0 \in \Theta$ is the initial guess of the unknown parameter vector $\theta$; $\tilde{x}_0$ is the initial guess of the unknown initial state $x_0$; the $(\sigma + n) \times (\sigma + n)$-dimensional matrix $\tilde{Q}_0$ is the quadratic weighting on the initial estimation error, quantifying the level of confidence in the estimate $[\tilde{x}_0', \tilde{x}_0']'$; $\tilde{Q}_0^{-1}$ admits the structure

$$\begin{bmatrix}
Q_0^{-1} & \Phi_0^{-1}
\Phi_0 Q_0^{-1} \Pi_0 + \Phi_0 \Phi_0 Q_0^{-1} \Phi_0'
\end{bmatrix}$$

where $Q_0$ and $\Pi_0$ are $\sigma \times \sigma$- and $n \times n$-dimensional positive-definite matrices, respectively.

The following notation will be used throughout this paper. Let $\tilde{x}$ denote the estimate of $x$, $\tilde{x}$ denote the state estimation error $x - \tilde{x}$, $\tilde{\theta}$ denote the estimate of $\theta$, $\tilde{\theta}$ denote the parameter estimation error $\theta - \tilde{\theta}$.

We intend to solve this robust adaptive control problem by formulating it as an $H^\infty$ control problem with imperfect state measurements. To do this, we first expand the state space to include the parameter $\theta$ as part of the state. Let $\xi$ denote the expanded state vector $\xi := [\theta', x']'$. Note that $\tilde{\theta} = 0$, we have the following expanded dynamics for system (2):

$$\dot{\xi} := \begin{bmatrix}
0_{\sigma \times \sigma} & 0_{\sigma \times n} \\
y \tilde{A}_{211} + u \tilde{A}_{212} & A
\end{bmatrix} \xi + \begin{bmatrix}
0_{\sigma \times 1} \\
B
\end{bmatrix} u + \begin{bmatrix}
0_{\sigma \times q} \\
D
\end{bmatrix} w := \tilde{A}(u, y) \xi + \tilde{B} u + \tilde{D} w$$

$$y = [u \tilde{C}_1 \ C] \xi + b_{p0} u + E w$$

$$:= \tilde{C}(u) \xi + b_{p0} u + E w$$  \hspace{1cm} (6b)
The worst-case optimization of the cost function (5) can be carried out in two steps as depicted in the following inequality:

\[
\sup_{(x_0, \theta, \hat{x}_d(t), \hat{y}_d(t)) \in \mathcal{W}'} J_{ttf} = \sup_{y(\infty) \in \mathcal{C}, y_d(\infty) \in \mathcal{C}} \sup_{(x_0, \theta, \hat{x}_d(t), \hat{y}_d(t)) \in \mathcal{W}'} (y(\infty), y_d(\infty)) \leq \sup_{y(\infty) \in \mathcal{C}, y_d(\infty) \in \mathcal{C}} \sup_{(x_0, \theta, \hat{x}_d(t), \hat{y}_d(t)) \in \mathcal{W}'} (y(\infty), y_d(\infty))
\]

By the cost-to-come function analysis of [23], we have

\[
\dot{\hat{\xi}} = (\hat{\hat{A}}(u, y) - \xi^2 \hat{L} \hat{C}(u)) \hat{\Sigma} + \hat{\Sigma} (\hat{\hat{A}}(u, y) - \xi^2 \hat{L} \hat{C}(u))' + \gamma^{-1} \hat{D} \hat{D}' - \gamma^{-2} \xi^2 \hat{L} \hat{L}'
\]

\[
- \hat{\Sigma} (\gamma^{-2} \xi^2 (\hat{C}(u))' \hat{C}(u) - (\hat{\hat{C}}(u))' \hat{C}(u) - \hat{\hat{Q}}(t, y[0, t], y_d[0, t])) \hat{\Sigma}, \quad \hat{\Sigma}(0) = \gamma^{-2} \hat{Q}_0^{-1}
\]

where \(\hat{\Sigma}\) is defined as \(\hat{\Sigma} = \{0_{1 \times \sigma}, L'/\\}\) and \(\xi_c := \xi - \hat{\hat{\xi}}\). Partition \(\hat{\Sigma}(t)\) as

\[
\hat{\Sigma}(t) = \begin{bmatrix} \Sigma(t) & \hat{\Sigma}_{12}(t) \\ \hat{\Sigma}_{21}(t) & \hat{\Sigma}_{22}(t) \end{bmatrix}
\]

\(\sigma \times \sigma\)-dimensional, and introduce \(\Phi(t) := \Sigma_{21}(t) (\Sigma(t))^{-1}\) and \(\Pi(t) := \gamma^2 (\Sigma_{22}(t) - \Sigma_{21}(t) (\Sigma(t))^{-1} \Sigma_{12}(t))\). Also partition \(\xi\) compatibly as \([\hat{\hat{x}}', x']\).

For the boundedness of \(\Sigma\), the weighting matrix \(\hat{\hat{Q}}(t, y[0, t], y_d[0, t])\) admits the following structure:

\[
\hat{\hat{Q}} = \hat{\Sigma}^{-1} \begin{bmatrix} 0_{\sigma \times \sigma} & 0_{\sigma \times n} \\ 0_{n \times \sigma} & \Delta(t) \end{bmatrix} \hat{\Sigma}^{-1} + \gamma (\hat{\hat{C}}(u(t)) + C \Phi(t))' (\gamma^{-2} \xi_c^2 - 1) (\hat{\hat{C}}(u(t)) + C \Phi(t)) + 0_{\sigma \times n} \begin{bmatrix} 0_{n \times \sigma} \\ 0_{n \times n} \end{bmatrix}
\]
where \( \Delta(t) = \gamma^{-2} \beta_{\Delta} \Pi(t) + \Delta_1 \), with \( \beta_{\Delta} \geq 0 \) being a constant and \( \Delta_1 \) being an \( n \times n \) positive-definite matrix, and \( \varepsilon \) is a scalar function defined by

\[
\varepsilon(t) := K_c^{-1} s_{\Sigma}(t) := \text{Tr}((\Sigma(t)^{-1})) / K_c, \quad t \in [0, \infty)
\]

or

\[
\varepsilon(t) := 1
\]

\( \text{To avoid the inversion of } \Sigma \text{ online, we define } s_{\Sigma}(t) := \text{Tr}((\Sigma(t)^{-1})) \), and its time derivative is given by

\[
\dot{s}_{\Sigma} = (\gamma^2 \gamma^2 - 1)(\tilde{C}_1 u + C \Phi)(\tilde{C}_1 u + C \Phi)
\]

\( s_{\Sigma}(0) = \gamma^2 \text{Tr}(Q_0) \)

Then, \( \varepsilon(t) = K_c^{-1} s_{\Sigma}(t) \), which does not require the inversion of \( \Sigma(t) \), when \( \varepsilon \) is defined by (10a).

Based on Lemma 1, we note that \( \gamma \geq \gamma^2 \). This means that the quantity \( \gamma^2 \) is the ultimate lower bound on the achievable performance level for the adaptive system, using the design method proposed in this section.

**Assumption 6**

If the matrix \( A - \gamma^2 LC \) is Hurwitz, then the desired disturbance attenuation level \( \gamma \geq \gamma^2 \). In case \( \gamma = \gamma^2 \), choose \( \beta_{\Delta} \geq 0 \) such that \( A - \gamma^2 LC + \beta_{\Delta} / 2 I_n \) is Hurwitz. If the matrix \( A - \gamma^2 LC \) is not Hurwitz, then the desired disturbance attenuation level \( \gamma > \gamma^2 \).

**Assumption 7**

The initial weighting matrix \( \Pi_0 \) in (11) is chosen as the unique positive-definite solution to the algebraic Riccati equation:

\[
(A - \gamma^2 LC + \beta_{\Delta} / 2 I_n) \Pi + \Pi (A - \gamma^2 LC + \beta_{\Delta} / 2 I_n) ^T \quad - \Pi C^T (\gamma^2 - \gamma^2) C \Pi(t) + DD^T \\
- \gamma^2 LL^T + \gamma^2 \Delta_1, \quad \Pi(0) = \Pi_0
\]

Then, we note that the unique positive-definite solution of (11) is time-invariant and equal to the initial value \( \Pi_0 \), and the matrix \( A_f := A - \gamma^2 LC - \Pi C^T (\gamma^2 - \gamma^2) C \) is Hurwitz.

To guarantee the boundedness of estimated parameters without persistently exciting signals, we introduce soft projection design on the parameter estimate, which is based on the a priori information on the bounds of the true value of the parameter vector \( \theta \), i.e. Assumption 4.

Define \( \rho := \inf (P(\theta)) / (0 \in \mathbb{R}^q \text{ and } b_{p_0} + \tilde{C}_1 \theta = 0) \), we have \( 1 < \rho \leq \infty \). And then we fix \( \rho \in (1, \rho) \), and define the open set \( \Theta_0 := \{ \theta \in \mathbb{R}^q | P(\theta) < p_{\rho_0} \} \). Our control design will guarantee the estimate \( \hat{\theta} \) always belongs to \( \Theta_0 \), which implies \( \tilde{b}_0 := b_{p_0} + \tilde{C}_1 \hat{\theta} \geq 0 \), for some positive constant \( c_0 \). Moreover, the convexity of \( P \) implies the following inequality \( (\tilde{C} P / c_0 \hat{\theta})(\theta - \hat{\theta}) < 0 \forall \hat{\theta} \in \mathbb{R}^q \setminus \Theta \). Set \( l_1(y_{[0,1]}), y_{[0,1]} = \tilde{\gamma}, I_2(y_{[0,1]}), \)}
we have the following 3
\[
\begin{align*}
\frac{\partial P}{\partial \theta} (\tilde{\theta}) \left( \right)^	op \forall \tilde{\theta} \in \Theta_o \setminus \Theta
\end{align*}
\] for \( \rho_o - P(\tilde{\theta}) \), and

\[
0_{\sigma \times 1}
\]

\[
\begin{align*}
P_r(\tilde{\theta}) := 
\left\{ \begin{array}{l}
\exp \left( \frac{1}{1-P(\tilde{\theta})} \right) \left( \frac{\partial P}{\partial \theta} (\tilde{\theta}) \right) ^\top \\
0_{\sigma \times 1}
\end{array} \right\} \forall \tilde{\theta} \in \Theta
\end{align*}
\]

\[
0
\]

where

\[
\begin{align*}
(A_f \tilde{y} - \bar{C} u) & + \bar{L} (y - \tilde{C} (u) \tilde{z}) \\
& + \bar{\Sigma} (12)
\end{align*}
\]

\[
\tilde{\xi}(0) = [\tilde{x}_0 \tilde{y}_0]^	op
\]

To simplify the controller structure, the dynamics for \( \Phi \) can be implemented with 3n integrators instead of the \( \sigma n \) integrators. First, we observe that the pair \( (A_f, C) \) is observable. Then we introduce the matrix \( M_f := [A_{f,1}^n p_n \ldots A_f p_n p_n] \), where \( p_n \) is an n-dimensional vector such that the pair \( (A_f, p_n) \) is controllable, which implies that \( M_f \) is invertible. Then the following 3n-dimensional prefiltering system for \( y \) and \( u \) generates the \( \Phi \) online:

\[
\begin{align*}
\dot{\eta} &= A_f \eta + p_n y, \quad \eta(0) = \eta_0 \quad (13a) \\
\dot{\lambda} &= A_f \lambda + p_n u, \quad \lambda(0) = \lambda_0 \quad (13b) \\
\dot{\lambda}_o &= A_f \lambda_o, \quad \lambda_o(0) = p_n \quad (13c)
\end{align*}
\]

\[
\Phi = [A_{f,1}^{n-1} \ldots A_f \eta \eta] M_f^{-1} \tilde{A}_{211} + [A_{f,1}^{n-1} \lambda \ldots A_f \lambda \lambda] M_f^{-1} \\
\times (\tilde{A}_{212} - \tilde{\xi}_2 L \tilde{C}_1 - \Pi (\tilde{\xi}_2 - \gamma^2) C' \tilde{C}_1) \\
+ [A_{f,1}^{n-1} \lambda_o \ldots A_f \lambda_o \lambda_o] M_f^{-1} \Phi_o
\]

(13d)

where

\[
\begin{align*}
\eta_0 & \in \mathbb{R}^\sigma, \lambda_0 \in \mathbb{R}^n, \text{ and } \Phi_o \in \mathbb{R}^{n \times \sigma} \text{ are such that}
\end{align*}
\]

(13d) holds at \( t = 0 \).
Based on inequality (7) in Section 3, the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms, 

\[
\sup_{(x_0, \theta, w_{\{0, \infty\}}, y_{\{0, \infty\}}) \in \mathcal{E}} J_{\pi f} \leq \sup_{y_{\{0, \infty\}} \in \mathcal{E}, \bar{y}_{\{0, \infty\}} \in \mathcal{E}} \left\{ \int_0^{t_f} \left( (C\dot{x}(\tau) + (b_p \theta + \bar{C}_1 \bar{\theta}(\tau))u(\tau) - y_d(\tau))^2 + |\xi(\tau)|^2\right) \right\}^{1/2}
\]

where the worst-case disturbance with respect to the value function \(W\) is given by, \(w_{\text{opt}} : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathbb{R} \times \mathcal{G} \to \mathbb{R}^q\),

\[
w_{\text{opt}}(\xi, \bar{\xi}, y_d, \bar{\Sigma}) = -\bar{\Sigma}^{-1}(\xi - \bar{\xi}) - \gamma^{-2} E'[\bar{\Sigma}(\xi - \bar{\xi})]\]

which holds as long as \(\Sigma > 0\) and \(\bar{\theta} \in \Theta_0\). Clearly, the closed-loop system is dissipative with storage function \(W\) and supply rate \(-(C \dot{x} + \bar{C}_1 \bar{\theta} u + b_p w - y_d)^2 + \gamma^2 |w|^2\).

This completes the control design step. We will turn to present the main results in the next section.

5. MAIN RESULT

With the estimation and control design of the previous section, the state of the closed-loop system is given by \(X := \{x', x', \bar{\Sigma}, s_\Sigma, \bar{\theta}, \bar{x}', \bar{\theta}'\}^T\), which belongs to the open set \(\mathcal{D} := \{X | s_\Sigma > 0, s_\Sigma > 0, \bar{\theta} \in \Theta_0\}\).

The dynamic for \(X\) is given by

\[
\dot{X} = F(X, y_d) + G(X)w, \quad X(0) = X_0
\]

where \(F\) and \(G\) are smooth mappings of \(\mathcal{D} \times \mathbb{R}\) and \(\mathcal{D}\), respectively; and the initial condition is

\[
X_0 \in \mathcal{D}_0 := \{X_0 \in \mathcal{D} | \theta \in \Theta, \bar{\theta} \in \Theta, \Sigma(0) > 0, s_\Sigma(0) = \gamma^2 \text{Tr}(Q_0) \leq K\}
\]
Since (20) holds, the value function \( W \) satisfies a Hamilton–Jacobi–Isaacs equation:

\[
\frac{\partial W}{\partial X}(X)F(X, y_d) + \frac{1}{4\gamma^2} \frac{\partial W}{\partial X}(X)G(X)(G(X))' \left( \frac{\partial W}{\partial X}(X) \right)' + Q(X, y_d) = 0, \quad \forall X \in \mathcal{X}, \quad \forall y_d \in \mathbb{R}
\]

where \( Q : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R} \) is smooth and given by

\[
Q(X, y_d) = |Cx + (b_{u0} + \tilde{C}_1 \theta)\mu(\hat{\theta}, \tilde{x}, y_d) - y_d|^2
\]

\[
+ |\xi - \tilde{\xi}|^2 \frac{\partial \Phi(\mu, \hat{\theta}, \tilde{x}, y_d), s_\Sigma}{\partial \mu}
\]

where \( \mu \) is defined by (19a), with the optimal choice (19b) for \( \tilde{\xi} \)

The closed-loop adaptive system possesses a strong robustness property, which will be stated precisely in the following theorem.

**Theorem 1**

Consider the robust adaptive control problem formulated in Section 3 with Assumptions 1–7 holding. The robust adaptive controller \( \mu \) defined by (19a), with the optimal choice (19b) for \( \tilde{\xi} \) achieving the following strong robustness properties for the closed-loop system.

1. Given \( c_w > 0 \) and \( c_d > 0 \), there exist a constant \( c_r > 0 \) and a compact set \( \Theta_r \subset \Theta \), such that for any uncertainty \( (x_0, \theta, \dot{w}(0, \infty), y_d(0, \infty)) \in \mathcal{W} \) with

\[
|x_0| \leq c_w, \quad |\dot{w}(t)| \leq c_d, \quad |\ddot{w}(t)| \leq c_w, \quad \forall t \in [0, \infty),
\]

all closed-loop state variables \( x, \tilde{x}, \hat{\theta}, \Sigma, s_\Sigma, \) and \( \Phi \) are bounded as follows, \( \forall t \in [0, \infty) \):

\[
|x(t)| \leq c_r, \quad |\tilde{x}(t)| \leq c_r, \quad |\hat{\theta}(t)| \leq c_r, \quad |\Phi(t)| \leq c_r
\]

\[
K_r^{-1} I_r \leq \Sigma(t) \leq K_r^{-1} I_r, \quad \gamma^2 \text{Tr}(Q_0) \leq s_\Sigma(t) \leq K_r
\]

Therefore, there is a compact set \( S \subseteq \mathcal{X} \) such that \( X(t) \in S \) for all \( t \in [0, \infty) \). Hence, there exists a constant \( c_0 > 0 \) such that \( |\mu(t)| \leq c_u, |\tilde{\xi}(t)| \leq c_u, |\tilde{\alpha}(t)| \leq c_u, |\tilde{\beta}(t)| \leq c_u, \) and \( |\alpha(t)| \leq c_u \).

2. The controller \( \mu \in \mathcal{M}_u \) achieves disturbance attenuation level \( \gamma \) for any uncertainty \((x_0, \theta, w(0, \infty), y_d(0, \infty)) \in \mathcal{W} \).

3. For any uncertainty \((x_0, \theta, \dot{w}(0, \infty), y_d(0, \infty)) \in \mathcal{W} \) with \( \dot{w}(0, \infty) \in L_2 \cap L_\infty, y_d(0, \infty) \in L_\infty, \) and \( y_d(0, \infty) \) being uniformly continuous, the output of the system, \( Cx + (\tilde{C}_1 \theta + b_{p0})u \), asymptotically tracks the reference trajectory, \( y_d \), i.e.

\[
\lim_{t \to +\infty} (Cx(t) + (\tilde{C}_1 \theta + b_{p0})u(t) - y_d(t)) = 0
\]

**Proof**

The following terms will be used throughout the proof and Appendix. Canonical form denotes Equation (78) in [23]. Bounding lemma denotes the Lemma 11 in [23].

For the first statement, fix \( c_w > 0 \) and \( c_d > 0 \), and consider any uncertainty \((x_0, \theta, \dot{w}(0, \infty), y_d(0, \infty)) \in \mathcal{W} \) that satisfies \( |x_0| \leq c_w, |w(t)| \leq c_w, |\dot{w}(t)| \leq c_d, |y_d(t)| \leq c_d, \forall t \in [0, \infty) \). We define \( [0, T_f] \) to be the maximal length interval on which the closed-loop system (22) has a solution that lies in \( \mathcal{X} \). By Lemma 1, we have \( \Sigma \) and \( s_\Sigma \) are upper and lower bounded as desired on \([0, T_f]) \).

Introduce the vector of variables \( X_e := (\hat{\theta}' (\tilde{x} - \Phi \tilde{\theta}'))' \), and two nonnegative and continuous functions defined on \( \mathbb{R}^{n+\sigma} \)

\[
U_M(X_e) := K_r |\hat{\theta}|^2 + \gamma^2 |\tilde{x} - \Phi \tilde{\theta}|^2
\]

\[
U_m(X_e) := \gamma^2 |\hat{\theta}|^2 + \gamma^2 |\tilde{x} - \Phi \tilde{\theta}|^2
\]

then, we have \( U_M(X_e) \leq \tilde{W}(t, X_e) \leq U_M(X_e) \), \( \forall t, X_e \in [0, T_f] \times \mathbb{R}^{n+\sigma} \), if we interpret \( W \) as \( \tilde{W}(t, X_e) \). Since \( U_m(X_e) \) is continuous, positive definite and radially unbounded, then \( \forall x \in \mathbb{R} \), the set \( S_{12} := \{ X_e \in \mathbb{R}^{n+\sigma} : U_m(X_e) < \varepsilon \} \) is compact or empty. Since \( |\dot{w}(t)| \leq c_w, \forall t \in [0, \infty) \), we have the following inequality for the derivative of \( W \):

\[
\tilde{W} \leq -\gamma^4 |x - \tilde{x} - \Phi (\hat{\theta} - \tilde{\theta})|^2
\]

\[
+ 2(\hat{\theta} - \tilde{\theta}) P_r(\hat{\theta}) + \gamma^4 |\tilde{x} - \Phi \tilde{\theta}|^2
\]

Since \( -\gamma^4 |x - \tilde{x} - \Phi (\hat{\theta} - \tilde{\theta})|^2 + 2(\hat{\theta} - \tilde{\theta}) P_r(\hat{\theta}) \) will tend to \(-\infty \) when \( X_e \) approaches the boundary of \( \Theta_r \times \mathbb{R}^n \), then there exists a compact set \( \Omega_1(c_w) \subset \Theta_r \times \mathbb{R}^n \), such that \( \tilde{W} < 0 \) for \( X_e \in (\Theta_r \times \mathbb{R}^n) \setminus \Omega_1 \). Note that \( X_e(t) \in (\Theta_r \times \mathbb{R}^n) \), \( \forall t \in [0, T_f] \), we have that for some \( c_1 \in \mathbb{R} \) and a compact set \( S_{12} \subseteq \mathbb{R}^{n+\sigma} \), \( \tilde{W}(t, X_e(t)) \leq c_1 \) and \( X_e(t) \subseteq S_{12}, \forall t \in [0, T_f] \). It follows that the signal \( X_e \) is bounded, i.e., \( \tilde{\theta} \) and \( \tilde{x} - \Phi \tilde{\theta} \) are bounded.
Based on the derivative of $\dot{x} - \Phi \dot{\theta}$, there is particular linear combination of the components of $\dot{x} - \Phi \dot{\theta}$, denoted by $\eta_L$,

$$d(\tilde{x}(t) - \Phi(t) \tilde{\theta}(t)) dt = A_f(\tilde{x}(t) - \Phi(t) \tilde{\theta}(t)) - \gamma^2 L \Pi C' y(t) + \gamma^2 L \Pi C' y_d(t) + (D + \gamma^2 L \Pi C' E) - \epsilon^2 \left( \Pi C' + L\right) \dot{M} \dot{w}(t)$$

$$\eta_L = TL(\tilde{x} - \Phi \tilde{\theta})$$

(26a)

(26b)

which is strictly minimum phase and has relative degree 1 with respect to $y$. Then by Lemma A.4, the composite system of (1) and (26) has relative degree 1 from input $u$ to output $\eta_L$; and it may serve as a reference system in the application of Bounding Lemma in the following proof.

Based on the dynamics of $\eta$, the relative degree for each element of $\eta$ is at least 1 with respect to the input $y$. Taking $\eta_L$ as the output and $u$ as the input of the reference system, we conclude $\eta$ is bounded by Bounding Lemma. Based on the dynamics of $\tilde{\lambda}$, the relative degree for each element of $\tilde{\lambda}$ is at least 1 with respect to the input $u$. Taking $\eta_L$ as the output and $u$ as the input of the reference system, we conclude $\tilde{\lambda}$ is also bounded.

Based on the dynamics of $\lambda_o$ (13c), we have $\lambda_o$ is bounded. Based on the formula for $\Phi$, (13d), we have $\Phi$ is bounded. Since $\tilde{x} - \Phi \tilde{\theta}, \Phi,$ and $\tilde{\theta}$ are bounded, we conclude that $\tilde{x}$ is bounded. Define the following equations to separate $x$ into two parts:

$$x = x_u + x_y$$

$$\dot{x}_u = A_f x_u + \left( B + \tilde{A}_{212} \dot{\theta} - (\tilde{C}_1 \theta + b_{p0}) \right) \times \left( \epsilon^2 L + \Pi C' \left( \epsilon^2 - \frac{1}{\epsilon^2} \right) \right) u$$

$$\dot{x}_y = A_f x_y + \left( \tilde{A}_{211} \dot{\theta} + \epsilon^2 L + \Pi C' \left( \epsilon^2 - \frac{1}{\epsilon^2} \right) \right) y + \left( D - \epsilon^2 L + \Pi C' \left( \epsilon^2 - \frac{1}{\epsilon^2} \right) E \right) \dot{M} \dot{w}$$

We observe that the relative degree for each element of $x_u$ is at least 1 with respect to the input $u$, and the relative degree for each element of $x_y$ is at least 1 with respect to the input $y$. Taking $\eta_L$ as the output and $u$ as the input of the reference system, we conclude $x_u$ is bounded by Bounding Lemma. Similarly, taking $\eta_L$ as the output and $y$ as the input of the reference system, we obtain that $x_y$ is bounded. Then, we have $x$ is bounded. Further, we have $\dot{x}$ is bounded, because $x$ and $\dot{x}$ are bounded and $\dot{x} = \dot{x} - \ddot{x}$. Since $\ddot{b}_0$ is bounded, and bounded away from 0, we conclude that $u$ is bounded. This further implies that $y$ is bounded.

Next, we need to prove the existence of a compact set $\Theta_o \subset \Theta$ such that $\tilde{\theta}(t) \in \Theta_o, \forall t \in [0, T_f]$.

Introduce the function $\gamma: [0, T_f] \times (\Theta_o \times \mathbb{R}^n) \rightarrow \mathbb{R}$

$$\gamma(t, x_e) := \tilde{W}(t, x_e) + (\rho_o - P(\tilde{\theta}))^{-1} P(\tilde{\theta})$$

We notice that, when $\tilde{\theta}$ approaches the boundary of $\Theta_o, P(\tilde{\theta})$ approaches $\rho_o$. Then $\gamma$ approaches $\infty$ as $x_e$ approaches the boundary of $\Theta_o \times \mathbb{R}^n$. We introduce two nonnegative and continuous functions defined on $\Theta_o \times \mathbb{R}^n$:

$$\gamma_M(x_e) := \Gamma_M(x_e) + (\rho_o - P(\tilde{\theta}))^{-1} P(\tilde{\theta})$$

$$\gamma_m(x_e) := \Gamma_m(x_e) + (\rho_o - P(\tilde{\theta}))^{-1} P(\tilde{\theta})$$

Then, by the previous analysis, we have

$$\gamma_m(x_e) \leq \gamma(t, x_e) \leq \gamma_M(x_e) \forall(t, x_e) \in [0, T_f] \times (\Theta_o \times \mathbb{R}^n)$$

Note that the set $S_{2\gamma} := \{x_e \in \Theta_o \times \mathbb{R}^n | \gamma_m(x_e) \leq \gamma \}$ is a compact set, $\forall \gamma \in \mathbb{R}$. Then, we consider the derivative of $\gamma$:

$$\dot{\gamma} = \tilde{W} + \rho_o(\rho_o - P(\tilde{\theta}))^{-2} \left( \frac{\partial P}{\partial \tilde{\theta}} \right) \dot{\theta}(t)$$

$$\leq -\gamma^2 |x - \dot{x} - \phi(\theta - \tilde{\theta})|_1^2 + 2(\theta - \tilde{\theta}) P_r(\tilde{\theta})$$

$$+ \gamma^2 \|M\|_2 \|w\|_2^2 - \rho_o \left( \frac{\partial P}{\partial \tilde{\theta}} \right) \left( \rho_o - P(\tilde{\theta}) \right)^{-2}$$

$$\times(K_c^{-1} P_r(\tilde{\theta})(\rho_o - P(\tilde{\theta}))^2 - 1) + c$$


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where \( c \geq 0 \) is a constant. Since \( \dot{Y} \) will tend to \(-\infty\) when \( X_e \) approaches the boundary of \( \Theta_o \times \mathbb{R}^n \), then there exists a compact set \( \Omega_2(c_w) \subset \Theta_o \times \mathbb{R}^n \), such that \( \forall X_e \in (\Theta_o \times \mathbb{R}^n) \setminus \Omega_2, \, \dot{Y} < 0 \). Then, \( Y(t, X_e(t)) \leq c_2 \), and \( X_e(t) \) belongs to the compact set \( S_{c_2} \subset \Theta_o \times \mathbb{R}^n, \forall t \in [0, T_f) \), for some \( c_2 \in \mathbb{R} \). Moreover, there exists a compact set \( \Theta_c \subset \Theta_o \), such that \( \dot{\theta}(t) \in \Theta_c, \forall t \in [0, T_f) \).

Then we can conclude that there exists a compact set \( S \subset \mathcal{D} \), such that \( X(t) \in S, \forall t \in [0, T_f) \). Therefore, it follows that \( T_f = \infty \) and \( X(t) \in S, \forall t \in [0, \infty) \).

For the second statement, we fix any uncertainty \((x_0, \theta, \dot{w}[0, \infty), y_d(0, \infty)) \in \hat{\mathcal{H}} \). Since \( y_d \) and \( \dot{w} \) are continuous functions, then, for any \( t_f > 0 \), there exist constants \( c_d \geq 0 \) and \( c_u \geq 0 \) such that \( |y_d(t)| \leq c_d \) and \( |\dot{w}(t)| \leq c_u \), \( \forall t \in [0, t_f] \). By the first statement, there exists a solution \( X: [0, t_f] \to \mathcal{D} \) for the closed-loop system (22). Then, by causality of system (22) and its smoothness, it implies that the closed-loop system (22) admits a unique solution on \([0, \infty)\). We set

\[
l(t, \theta, x, y_d[0,t]) := |x - \hat{x}(t) - \Phi(t)(\theta - \hat{\theta}(t))|^2_{\Pi^{-1} \Delta \Pi^{-1}} \]

\[
-2(\theta - \hat{\theta}(t))^\top P_s(\hat{\theta}(t)) + \varepsilon(\eta_2^2 - 1) \]

\[
\times |\theta - \hat{\theta}(t)|^2_{(u(t)\hat{C}_1 + C\Phi(t))' (u(t)\hat{C}_1 + C\Phi(t))} \]

Then, we have

\[
\sup_{(x_0, \theta, w[0,\infty), y_d[0,\infty]) \in \hat{\mathcal{H}}} J_{t_f} \leq \sup_{(x_0, \theta, w[0,\infty), y_d[0,\infty]) \in \hat{\mathcal{H}}} \left\{ J_{t_f} + \int_0^{t_f} \tilde{W} d\tau - W(X(t_f)) + W(X(0)) \right\} \]

\[
\leq \sup_{(x_0, \theta, w[0,\infty), y_d[0,\infty]) \in \hat{\mathcal{H}}} \left\{ -\gamma^2 |w(\tau) - w_{opt}(\tau)|^2 d\tau - W(X(t_f)) \right\} \]

This establishes the second statement.

For the last statement, we consider the following inequality:

\[
\int_0^{t_f} \tilde{W} d\tau \leq \int_0^{t_f} (-|C x(\tau) + (\tilde{C}_1 \theta + b_{p0}) u(\tau) - y_d(\tau)|^2)
+ \gamma^2 |\dot{M} \dot{w}(\tau)|^2 d\tau \quad \forall t_f \geq 0 \]

Then,

\[
\int_0^{\infty} |C x(\tau) + (\tilde{C}_1 \theta + b_{p0}) u(\tau) - y_d(\tau)|^2 d\tau
\leq \int_0^{\infty} (\eta^2 |\dot{M} \dot{w}(\tau)|^2) d\tau + W(X(0)) < +\infty \]

Since \( y_d - C \tilde{x} \) is uniformly continuous and bounded, \( \tilde{C}_1 \theta + b_{p0} \) is uniformly continuous and bounded away from 0, then \( u \) is uniformly continuous. Then, \( C x + (\tilde{C}_1 \theta + b_{p0}) u - y_d \) is uniformly continuous. It further implies that

\[
\lim_{t \to +\infty} (C x(t) + (\tilde{C}_1 \theta + b_{p0}) u(t) - y_d(t)) = 0 \]

This completes the proof of the theorem. \( \square \)

6. EXAMPLE

In this section, we present one example to illustrate the main results of this paper. The design was carried out using MATLAB™ symbolic computation tools, and the closed-loop system was simulated using SIMULINK™.

As a simple example of the inaccurate implementation of controller, consider the circuit shown in Figure 1. \( v_i \) is the theoretical input voltage; \( v_o \) is the actual control input to the physical plant; \( b_0 \) is the unknown ratio of \( v_i \) to \( v_o \); \( v_e \) is an exogenous bias sinusoidal disturbance signal with unknown bias, unknown magnitude, unknown frequency, and unknown phase, and we set \( v_e \) to \( 2 + \frac{1}{3} \sin(2t + \arcsin(2\sqrt{5}/5)) + \cos(2t + \pi/4) \) for simulation purpose; \( v_w \) is an exogenous noise signal, and we assume we have some a priori knowledge of \( v_w \) and which can be modeled as:

\[
\dot{x}_4 = v_e - \dot{x}_4 - \dot{x}_5 + v_i + \dot{w}_1 \quad \dot{x}_4(0) = 0
\]

\[
\dot{x}_5 = 2\dot{x}_4 - 2\dot{w}_2 \quad \dot{x}_5(0) = 0
\]

\[
v_w = \dot{x}_4 + \dot{x}_5 + \dot{w}_3
\]

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where \( \dot{w}_1, \dot{w}_2, \) and \( \dot{w}_3 \) denote the uncertainties of the modeling and the arbitrary disturbance in the measurement channel. The objective is to achieve asymptotic tracking of \( v_e \) to the reference trajectory \( y_d \).

The equation that describes the circuit is obtained as
\[
y = v_e + b_0 u + \dot{w}
\]
where \( u = v_i, y = v_o, \dot{w} = v_w, \) and \( b_0 \) is assumed to be unknown and belong to the interval \([0.2, 1]\), whose true value is set to 1 for illustration purposes. \( v_e \) can be modeled as the output of a third-order linear system as the following:
\[
\begin{align*}
\dot{x}_1 &= \dot{x}_3; \quad \dot{x}_1(0) = 2 + \sqrt{5}/5 + \sqrt{2}/2 \\
\dot{x}_2 &= \dot{x}_3; \quad \dot{x}_2(0) = \sqrt{5}/5 + \sqrt{2}/2 \\
\dot{x}_3 &= \tilde{\theta}_1 \dot{x}_2; \quad \dot{x}_3(0) = \sqrt{5}/5 - \sqrt{2}/2
\end{align*}
\]
where \( \tilde{\theta}_1 \) is the negative square of the sinusoid frequency. It is assumed to be unknown and belong to the interval \([-4, 0]\), and its true value is set to \(-4\) for illustration purposes. The initial condition of \( \dot{x}_1(0), \dot{x}_2(0), \) and \( \dot{x}_3(0) \) determines the bias, magnitude, and phase of sinusoid signal, whose true values are set for simulation purpose. Note that the true system satisfies Assumptions 1 and 2.

To normalize the parameters, we set \( b_{\rho_0} = 0.6 \), and define \( \theta = [\theta_1, \theta_2, \theta_3]' \), where \( \theta_1 = (\tilde{\theta}_1 + 2)/2, \theta_2 = (b_0 - b_{\rho_0})/0.4, \) and \( \theta_3 = \theta_1 \theta_2. \) Then, the true value for parameter vector \( \theta \) is \([-1 1 -1]' \), and \( \theta_1, \theta_2, \) and \( \theta_3 \) all belong to the interval \([-1, 1]\).

Introduce the state transformation and disturbance transformation as follows:
\[
x = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
2 & 0 & 1 & 2 & 0 \\
-\theta_1 + 4 & \theta_1 & 2 & -\theta_1 & -\theta_1 \\
-2\theta_1 & 2\theta_1 & 4 & -2\theta_1 & 0 \\
-4\theta_1 & 4\theta_1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5
\end{bmatrix}
\]
\[
w = \frac{1}{8}
\begin{bmatrix}
1 & -2 & 0 \\
2 & 0 & -2\theta_1 \\
1 - \theta_1 & 2\theta_1 - 2 & -\theta_1 \\
2 - 2\theta_1 & 0 & -2\theta_1 \\
0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3
\end{bmatrix}
\]
we obtain the design model for the adaptive controller
\[
\dot{x} = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
-4 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 1 & 0 \\
-4 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
x + y
\begin{bmatrix}
2\theta_1 \\
4\theta_1 \\
0 \\
0 \\
0
\end{bmatrix}
\]
\[
+ u
\begin{bmatrix}
0 \\
-1.2\theta_1 & -0.8\theta_3 \\
-3.2\theta_1 & -0.8\theta_3 \\
-6.4\theta_1 & -1.6\theta_3 \\
0
\end{bmatrix}
\begin{bmatrix}
8 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 & 0 \\
0 & 0 & 16 & 0 & 0 \\
0 & 0 & 0 & 16 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
w
\]
\[
y = [1 0 0 0 0] x + 0.6 u + 0.4 \theta_2 u + [0 0 0 0 2] w
\]

For the adaptive control design, the ultimate lower bound for the achievable disturbance attenuation level is \( 2 \) with respect to \( w \). We set the desired disturbance attenuation level \( \gamma = 10 \). The convex function \( P(\theta) \) is chosen as \( P(\theta) = 0.8\theta_1^2 + 0.1(\theta_2^2 + \theta_3^2) \).
For other design and simulation parameters, we select

\[
\begin{bmatrix}
\dot{x}_0 &= [0 \ 0 \ 0 \ 0]^	op, & \dot{\theta}_0 &= [0 \ 0]^	op \\
Q &= [0.001 I_3, & \Lambda &= I_5, & p_n &= [0 \ 0 \ 0 \ 1]^	op \\
\Phi &= [0_{5 \times 3}, & K_c &= 0.3 \ \rho_o &= 1.25, & \beta_A &= 0 \\
e &= K_c^{-1} s \Sigma, & \eta_0 &= [0_{5 \times 1}, & \lambda_0 &= [0_{5 \times 1}]
\end{bmatrix}
\]

Set the reference trajectory, \( y_d = \sqrt{\sin(3t)} \), which is uniformly continuous on \([0, \infty)\). We present two sets of simulation results for this example. The first set is to illustrate the regulatory behavior of the adaptive controller. We set \( \dot{w}(t) \equiv 0 \). The results are shown in Figure 2. We observe that the parameter estimates converge to their true values, and the tracking error converges to 0. The transient of the system is well-behaved, and the control magnitude is upper bounded by 3.7. The convergence rate is clearly exponential. When \( t = 250 \), the magnitude of the tracking error reduces to \( 10^{-15} \), and magnitude of the parameter estimation error reduces to \( 10^{-14} \). This set of result also demonstrates the asymptotic rejection of the unknown biased sinusoidal distance signal \( v_e \), and \( v_w \) if it can be actually modeled.

The second set of simulation results is to illustrate the robustness property of the adaptive controller. We set the disturbances as

\[
\dot{w} = \begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3
\end{bmatrix} = \begin{bmatrix}
0.4 \sin \left( 2t + \frac{3}{\pi} \right) + \text{Sampled white noise with power 0.08 and sample rate 1 HZ} \\
0.8 \sin \left( 3t + \frac{4}{\pi} \right) + \text{Sampled white noise with power 0.08 and sample rate 1 HZ} \\
0.5 \sin \left( 4t + \frac{5}{\pi} \right) + \text{Sampled white noise with power 0.08 and sample rate 1 HZ}
\end{bmatrix}
\]

The simulation results are shown in Figure 2. We observe that the parameter estimates no longer converge to the true values, but the output tracking error satisfies the targeted attenuation level as desired according to Figure 3(f). The transient of the system is well-behaved, and the control magnitude is upper bounded by 4.

7. CONCLUSIONS

In this paper, we studied the adaptive control design for tracking and disturbance attenuation for SISO linear systems with zero relative degree under noisy output measurements. We assume that the linear system has a known upper bounds (which is greater than zero) of the dynamic order, has a strictly minimum phase transfer function with known relative degree 0. We allow the system to be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov and those uncontrollable modes on the \( j\omega \)-axis are uncontrollable from the disturbance input. Under these assumptions, the system may be transformed into the design model, which is linear in all of the unknown quantities. The objectives of the control design are to make the noiseless output of the system to asymptotically track the reference trajectory, and guarantee the boundedness of all closed-loop signals, while rejecting the uncertainties in the system, which comprises the initial state of the system, the true value of the unknown parameter vector, the waveform of the exogenous disturbance input, and the reference trajectory. We use \( H^\infty \)-optimal control formulation and game-theoretic approach to derive the robust adaptive controller. We treat the unknown parameter vector as part of the expanded state vector, and formulate this adaptive control problem as a nonlinear \( H^\infty \)-optimal control problem with imperfect state measurements. For the design model, we assume that the measurement channel is noisy, such that the estimation step is a nonsingular optimization problem. We further assume that the unknown parameter vector belongs to a convex compact set characterized by a known smooth nonnegative radially unbounded and strictly convex function \( P(\theta) \). Furthermore, for any parameter vector belonging to the set, the corresponding high-frequency
Figure 2. System response under reference trajectory $y_d(t) = \sqrt{|\sin(3t)|}$: (a) tracking error; (b) tracking error (long term); (c) parameter estimates; solid line for $\hat{\theta}_1$; dash line for $\hat{\theta}_2$; and dash dot line for $\hat{\theta}_3$; (d) parameter estimation error (long term); solid line for $\hat{\theta}_1 - \hat{\theta}_1$; dash line for $\hat{\theta}_2 - \hat{\theta}_2$; and dash dot line for $\hat{\theta}_3 - \hat{\theta}_3$; (e) control input; and (f) performance index $\int_0^T ((Cx + b_0 u - y_d)^2 - \gamma^2 |w|^2) \, dt$. 

Figure 3. System response under reference trajectory $y_d(t) = \sqrt{|\sin(3t)|}$: (a) tracking error; (b) control input; (c) parameter estimates; solid line for $\hat{\theta}_1$; dash line for $\hat{\theta}_2$; and dash dot line for $\hat{\theta}_3$; (d) state estimates; solid line for $\hat{x}_1$; dash line for $\hat{x}_2$; and dash dot line for $\hat{x}_3$; long dash line for $\hat{x}_4$; thick dash line for $\hat{x}_5$; (e) disturbances; solid line for $\hat{w}_1$; dash line for $\hat{w}_2$; and dash dot line for $\hat{w}_3$; and (f) performance index $\int_0^1 ((Cx + b_0 u - y_d)^2 - \gamma^2 |w|^2) \, dt$. 

gain is never zero. Then, the *cost-to-come* function analysis is applied to derive the worst-case identifier and state estimator, which have a finite-dimensional structure. Using *a priori* information on the parameter vector, a smooth soft projection algorithm is applied in the estimation step, which relieves the persistency of excitation condition for the closed-loop system. Then, the closed-loop system is robust with or without the persistently exciting signals. After the estimation step is completed, the original problem becomes a nonlinear $H^\infty$-optimal control problem under full-information measurements. Then, the controller can be obtained directly from the cost function in one step. The controller then achieve the desired disturbance attenuation level, with the ultimate lower bound of the attenuation level being the noise intensity in the measurement channel. It guarantees the boundedness of all closed-loop signals and achieves asymptotic tracking of uniformly continuous reference trajectories when the disturbance is of finite energy and bounded. Because of the assumptions we made on the unknown system, the adaptive controller can asymptotically cancel out the effect of exogenous sinusoidal inputs with unknown magnitudes, phases, and frequencies, as long as we extend our system model to incorporate the knowledge of the existence of such sinusoidal inputs. This property of our adaptive controller has significant impact on active noise cancelation applications. This feature is illustrated by a numerical example, which corroborates all of our theoretical findings.

**APPENDIX A: FOUR LEMMAS**

**Lemma A.1**
 Define $\rho:=\inf\{P(\tilde{\theta})|\tilde{\theta}\in\mathbb{R}^\sigma$ and $b_{p0}+\tilde{C}_1\tilde{\theta}=0\}$, with Assumption 4 holding. Then $1<\rho\leq\infty$. Fix any $\rho_o \in (1, \rho)$, and define the open set $\Theta_o:=\{\tilde{\theta}|P(\tilde{\theta})<\rho_o\}$. For any $\tilde{\theta}\in\Theta_o$, it implies $|\tilde{b}_0|>c_0>0$, where $\tilde{b}_0:=b_{p0}+\tilde{C}_1\tilde{\theta}$, for some positive constant $c_0$.

**Proof**
 First we show $1<\rho\leq\infty$. Denote $\Theta_p:=\{\tilde{\theta}\in\mathbb{R}^\sigma|b_{p0}+\tilde{C}_1\tilde{\theta}=0\}$. There are two exhaustive and mutually exclusive cases.

**Case 1:** $\Theta_p=\emptyset$. It immediately implies $\rho=\inf\{P(\tilde{\theta})|\tilde{\theta}\in\Theta_p\}=+\infty$.

**Case 2:** $\Theta_p\neq\emptyset$. Then there exists a $\theta_0\in\Theta_p$ such that $b_{p0}+\tilde{C}_1\theta_0=0$. Take $a=P(\theta_0)$ and denote

$$\Theta_\tilde{\theta}:=\{\tilde{\theta}\in\mathbb{R}^\sigma|b_{p0}+\tilde{C}_1\tilde{\theta}=0 \text{ and } P(\tilde{\theta})\leq a\}$$

Since $\inf\{P(\tilde{\theta})|\tilde{\theta}\in\Theta_\tilde{\theta}\} < \inf\{P(\tilde{\theta})|\tilde{\theta}\in\Theta_p \cap \Theta_\tilde{\theta}\}$, we have $\rho=\inf\{P(\tilde{\theta})|\tilde{\theta}\in\Theta_p\}$. By Assumption 4, we have set $\Theta_\tilde{\theta}$ as a compact set in $\mathbb{R}^\sigma$. Then, there exists a $\theta_1\in\Theta_\tilde{\theta}$ such that $\rho=P(\theta_1)$. By the definition of set $\Theta$, we have $\theta_1\notin\Theta$. Then, $P(\theta_1)>1$, i.e. $\rho>1$. Combining the Cases 1 and 2, we have $1<\rho<\infty$.

Next, we show when we fix a $\rho_o\in(1, \rho)$ and define $\Theta_o:=\{\tilde{\theta}\in\mathbb{R}^\sigma|P(\tilde{\theta})<\rho_o\}$, there exists a $c_0$ such that $|\tilde{b}_0|>c_0>0$ for any $\tilde{\theta}\in\Theta_o$. Define $\Theta_o:=\{\tilde{\theta}\in\mathbb{R}^\sigma|P(\tilde{\theta})<\rho_o\}$. We claim that $\Theta_o\cap\Theta_p=\emptyset$. It is obvious when one or both of the above sets are empty. Consider none of the sets is empty and suppose $\theta_2$ is an element of $\Theta_o\cap\Theta_p$. Then, we have $\rho_o\geq P(\theta_2)>\rho$, which contradict $1<\rho_o<\rho$. This implies such $\theta_2$ does not exist.

We note $\Theta_o$ is a nonempty compact and convex set, and take $c_0=\inf_{\tilde{\theta}\in\Theta_o}\text{sgn}(b_0)(b_{p0}+\tilde{C}_1\tilde{\theta})$. Then, there exists a $\tilde{\theta}_0\in\Theta_o$ such that $c_0=\text{sgn}(b_0)(b_{p0}+\tilde{C}_1\tilde{\theta}_0)$. We claim $c_0$ is the positive constant we derive. Suppose $c_0<0$.

**Case 1:** $c_0=0$, i.e. $b_{p0}+\tilde{C}_1\tilde{\theta}_0=0$. Then, $\tilde{\theta}_0\in\Theta_p$. It contradicts $\Theta_o\cap\Theta_p=\emptyset$.

**Case 2:** $c_0<0$. Note that $\theta\in\Theta_o\subseteq\Theta_o$. By the definition of $\Theta_o$, we have $\text{sgn}(b_0)(b_{p0}+\tilde{C}_1\tilde{\theta}_0)>0$. There exists $\varepsilon\in(0,1)$ such that $\text{sgn}(b_0)(b_{p0}+\tilde{C}_1\tilde{\theta}_0)>\varepsilon$. Then, there exists $\tilde{\theta}_1=\tilde{\theta}_0+\varepsilon\tilde{\theta}_0+(1-\varepsilon)\theta_0$. Denote $\tilde{\theta}_1=\tilde{\theta}_0+(1-\varepsilon)\theta_0$, and by the convexity of $\Theta_o$, we obtain $\tilde{\theta}_1\in\Theta_o$. But $b_{p0}+\tilde{C}_1\tilde{\theta}_1=0$ implies $\tilde{\theta}_1\in\Theta_p$. It contradicts $\Theta_o\cap\Theta_p=\emptyset$. Then, we have obtained contradiction for all cases. Hence, $c_0>0$. $\forall \tilde{\theta}\in\Theta_o$, $\text{sgn}(b_0)(b_{p0}+\tilde{C}_1\tilde{\theta})\geq c_0=\text{sgn}(-b_{p0}+\tilde{C}_1\tilde{\theta}).$ Since $|\tilde{b}_0|\leq|\text{sgn}(b_0)(b_{p0}+\tilde{C}_1\tilde{\theta})|$, for any $\tilde{\theta}\in\Theta_o$, we obtain $|\tilde{b}_0|>c_0>0$. This completes the proof.

The following lemma is used to show that the true system (1) can be expanded to a system, which is of its upper bound dimension, and satisfies Assumption 2.


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Lemma A.2

Consider system (1) under Assumption 2. Assume \( \dot{n} < n \). Then, we can always add additional \((n - \dot{n})\)-dimensional dynamics, such that the expanded system is of order \( n \) and satisfies Assumption 2. Furthermore, the expanded system admits the same mapping from \((\hat{x}_0, u_{[0, \infty)}, \hat{w}_{[0, \infty)})\) to \( y_{[0, \infty)}\) as system (1).

Proof

We will discuss two exhaustive and mutually exclusive cases: Case 1: \( \dot{n} = 0 \); Case 2: \( \dot{n} > 0 \). First, consider Case 1: \( \dot{n} = 0 \). We expand the system as follows:

\[
\begin{align*}
\dot{\tilde{x}} &= \tilde{A}_{22} \tilde{x}, \quad \tilde{x}(0) = 0 \quad (A1a) \\
y &= \tilde{C} \tilde{x} + b_0 u + \tilde{E} \hat{w} \quad (A1b)
\end{align*}
\]

where \( \tilde{x} \) is the \( n \)-dimensional state vector; \( \tilde{A}_{22} \) is Hurwitz, and \((\tilde{A}_{22}, \tilde{C})\) is observable. We notice that, since \((A1a)\) is a stable system with zero initial conditions, the trajectory \( \tilde{x} \) will always be zeros. Then, the trajectory \( y \) will remain the same for the expanded system \((A1a)\) and the original system (1) when the trajectories for \( u \) and \( \hat{w} \) remain unchanged. Clearly, \((A1a)\) satisfies Assumption 2.

Next, we consider Case 2: \( \dot{n} > 0 \). Since system (1) satisfies Assumption 2, we assume it is given in observer canonical form. Then, we expand the original system to \( n \) dimensional by adding the \( \tilde{x} \) dynamics, which is \((n - \dot{n})\) dimensional,

\[
\begin{align*}
\begin{bmatrix}
\dot{x} \\
\dot{\tilde{x}} \\
\end{bmatrix} &=
\begin{bmatrix}
\hat{A} & \hat{A}_{12} \\
0 & \hat{A}_{22}
\end{bmatrix}
\begin{bmatrix}
x \\
\tilde{x}
\end{bmatrix} +
\begin{bmatrix}
\hat{B} \\
0
\end{bmatrix} u +
\begin{bmatrix}
\hat{D} \\
0
\end{bmatrix} \hat{w} \quad (A2a)
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\dot{x}(0) \\
\dot{\tilde{x}}(0)
\end{bmatrix} &=
\begin{bmatrix}
x_0 \\
0
\end{bmatrix} \\
y &= [\tilde{C} \ 0] [\dot{\tilde{x}} \ \dot{x}'] + b_0 u + \hat{E} \hat{w} \quad (A2b)
\end{align*}
\]

where \( \hat{A}_{12} = [0 \ -\hat{C}]' \tilde{C} \) is a row vector, \( \hat{A}_{22} \) is Hurwitz, and the pair \((\hat{A}_{22}, \hat{C})\) is observable.

We observe that the trajectories \( \dot{x} \) and \( y \) in \((A2)\) are as same as those in (1), when \( \dot{x}_0, u \) and \( \hat{w} \) remain unchanged. It is easy to check that \((A2)\) is observable because of the observability of the pair \((\hat{A}_{22}, \hat{C})\). Since the expanded dynamics, \( \dot{\tilde{x}} \), is uncontrollable from \( u \), the transfer function of \((A2)\) is the same one of (1), and it is a strictly minimum phase with relative degree 0.

To check other conditions of Assumption 2, we follow the derivation below.

First, we transform system (1) into Jordan canonical form representation. We choose an invertible complex matrix \( T_J \). The transformation \( \dot{x} = T_J z \) will transform the dynamics (1) into

\[
\begin{align*}
\dot{z} &= \text{diag}(J_1, \ldots, J_k) + [B'_{j1} \ \cdots \ B'_{jk}]' u \\
&\quad + [D'_{j1} \ \cdots \ D'_{jk}]' \hat{w} \quad (A3a)
\end{align*}
\]

\[
\begin{align*}
y &= [C_{J1} \ \cdots \ C_{jk}] z + b_0 u + \hat{E} \hat{w} \quad (A3b)
\end{align*}
\]

where \( J_i \) is a Jordan block associated with eigenvalue \( \lambda_i, i = 1, \ldots, k, k \in \mathbb{N} \); in the column vectors of \( T_J \), complex conjugate pairs appear together. Since a single-output Jordan canonical form state-space representation is observable only if there is only one Jordan block associated with each distinct eigenvalue, then \( i \neq j \Rightarrow \lambda_i \neq \lambda_j, \forall i, j \in \{1, \ldots, k\} \). Partition \( T_J = [T_{J1} \ \cdots \ T_{Jk}] \) accordingly. For any \( i, j \in \{1, \ldots, k\} \), \( \lambda_i \) is the complex conjugate of \( \lambda_j \), implies that \( T_{Ji} \) is the complex conjugate of \( T_{Jj} \). By Assumption 2, if \( \lambda_i \) is any uncontrollable mode from \( u \), then the last row of \( B_i \) is zero; in addition, if \( \text{Re} (\lambda_i) = 0 \), then the last row of \( D_i \) is zero. When \( \text{Re} (\lambda_i) = 0 \) and \( \lambda_i \) is an uncontrollable mode, the second to last row of \( B_{ji} \) must be nonzero, otherwise, the uncontrollable part of (1) with respect to \( u \) will contain a Jordan block associated with \( \lambda_i \) of order at least 2, which is not stable in the sense of Lyapunov. Next, we separate the above Jordan Canonical form into controllable and uncontrollable parts with respect to \( u \), and we denote them as \( z_c \) and \( z_e \), respectively. Then, we separate the uncontrollable part \( z_e \) into \( \text{Re} (\lambda) < 0 \) and \( \text{Re} (\lambda) = 0 \) parts, and we denote them as \( z_3 \) and \( z_4 \), respectively. This corresponds to the existence of a permutation matrix \( T_c \), and the transformation

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where $z_c$ is the $n_1$-dimensional controllable part; $z_3$ and $z_4$ are $n_2$- and $n_3$-dimensional, respectively, and compose the uncontrollable part; $n_i \in \mathbb{N} \cup \{0\}$, $i=1,2,3$. The matrix $T_c$ can be taken such that $\hat{A}_c$ = block diagonal $(J_{c1}, \ldots, J_{ck})$, $\hat{A}_{33}$ = block diagonal $(J_{31}, \ldots, J_{3k})$, $\hat{A}_{44}$ = block diagonal $(\hat{J}_{c1}, \ldots, \hat{J}_{ck})$, where $0 \leq k_1$, $k_2$, $k_3 \leq k$; $J_{ci}$, $i=1, \ldots, k_1$, $J_{3j}$, $j=1, \ldots, k_3$, are Jordan blocks associated with eigenvalues $\lambda_{ci}$, $i=1, \ldots, k_1$, $\lambda_{3j}$, $j=1, \ldots, k_3$, respectively. $\lambda_{ci} \neq \lambda_{cj}$, if $i \neq j$; $\lambda_{3i} \neq \lambda_{3j}$, if $i \neq j$; and $\lambda_{4i} \neq \lambda_{4j}$, if $i \neq j$. Furthermore, $\forall i \in \{1, \ldots, k_3\}$, $\hat{J}_{ci}$ is of order $1$, and $\lambda_{4i}$ has zero real part; $\forall i \in \{1, \ldots, k_2\}$, $\lambda_{3i}$ has negative real part. Complex conjugate pairs appear together in each of the set $\{\lambda_{c1}, \ldots, \lambda_{ck}\}$, $\{\lambda_{31}, \ldots, \lambda_{3k}\}$, $\{\lambda_{41}, \ldots, \lambda_{4k}\}$. Partition $T_{c} = [T_{c1} \cdots T_{cck} T_{c31} \cdots T_{c3k2} T_{c41} \cdots T_{c4k3}]$ accordingly. For any $i, j \in \{1, \ldots, k_1\}$, $\hat{\lambda}_{ci}$ is the complex conjugate of $\hat{\lambda}_{cj}$, implies $T_{cij}$ is the complex conjugate of $T_{cji}$. Similar statements can be made for $\lambda_{3i}$ and $\lambda_{3j}$, and also for $\lambda_{4i}$ and $\lambda_{4j}$.

It is easy to see that, we can choose three complex invertible matrices $T_{c1}, T_{c3}, T_{c4}$, such that $T_f$ = block diagonal $(T_{c1}, T_{c3}, T_{c4})$, whose partitioning is compatible with that of $[z_2'| z_3' | z_4']^T$ and $T_1 = T_c T_c^T$ is a real invertible matrix. Then, the coordinate transformation $\tilde{x} = T_1[x_2' x_3' x_4']^T$ will transform (1) into

$$
\begin{bmatrix}
\dot{\tilde{x}}_c \\
\dot{\tilde{x}}_3 \\
\dot{\tilde{x}}_4
\end{bmatrix} =
\begin{bmatrix}
A_c & A_{c,34} & \tilde{\hat{A}}_c \\
0_{(n-n_1) \times n_1} & \hat{A}_{33} & 0 \\
0 & \hat{A}_{44} & 0
\end{bmatrix}
\begin{bmatrix}
x_c \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix}
+ \begin{bmatrix}
B_c \\
0 \\
0
\end{bmatrix} u
+ \begin{bmatrix}
D_c \\
D_3 \\
0
\end{bmatrix} \dot{w}
$$

(A5a)

$$
y = [C_c \tilde{\hat{C}}_3 \tilde{\hat{C}}_4][x_2'| \tilde{x}_3' | \tilde{x}_4']^T + b_0 u + \tilde{E} \dot{w}
$$

(A5b)

where all matrices are real, the triple $(A_c, B_c, C_c)$ is controllable and observable, $\hat{A}_{33}$ is Hurwitz, $\hat{A}_{44}$ has all eigenvalues lying on the $j\omega$-axis and all Jordan blocks being of order 1.

Then, (A2) admits the following state space representation:

$$
\begin{bmatrix}
\dot{x}_c \\
\dot{\tilde{x}}_3 \\
\dot{\tilde{x}}_4 \\
\dot{\tilde{x}}_\bar{c}
\end{bmatrix} =
\begin{bmatrix}
A_c & \hat{A}_{c,34} & \tilde{\hat{A}}_c \\
0_{(n-n_1) \times n_1} & \hat{A}_{33} & 0 \\
0 & \hat{A}_{44} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_c \\
\tilde{x}_3 \\
\tilde{x}_4 \\
\tilde{x}_\bar{c}
\end{bmatrix}
+ \begin{bmatrix}
B_c \\
0 \\
0
\end{bmatrix} u
+ \begin{bmatrix}
D_c \\
D_3 \\
0
\end{bmatrix} \dot{w}
$$

(A6a)

$$
y = [C_c \tilde{\hat{C}}_3 \tilde{\hat{C}}_4][x_2'| \tilde{x}_3' | \tilde{x}_4']^T + b_0 u + \tilde{E} \dot{w}
$$

(A6b)
where 
\[
\begin{bmatrix}
\tilde{x}_3 \\
\tilde{z}_o
\end{bmatrix} = \begin{bmatrix}
\tilde{x}_3 \\
\tilde{z}_o
\end{bmatrix}
\text{ and } \tilde{A}_{33} = \begin{bmatrix}
\dot{A}_{33} & \tilde{A}_3 \\
0 & \tilde{A}
\end{bmatrix}
\]

Since \(\dot{A}_{33}\) and \(\tilde{A}\) are Hurwitz, then \(\tilde{A}_{33}\) is Hurwitz. Based on (A6), clearly the expanded system satisfies Assumption 2.

This complete the proof of the lemma. \(\square\)

We will use the following lemmas to give sufficient condition on a linear system that may serve as a reference system in the application of Bounding Lemma.

**Lemma A.3**
Consider a linear time-invariant system with the following state-space representation:
\[
\begin{align*}
\dot{x} &= Ax + Bu + Dw \\
y &= Cx + Ew
\end{align*}
\]

where \(x\) is the \(n\)-dimensional state vector, \(n \in \mathbb{N}\); \(w\) is \(q\)-dimensional disturbance input, \(q \in \mathbb{N}\); \(u\) is the scalar input, and \(y\) is the scalar output; \(A\) is Hurwitz; and the transfer function from \(u\) to \(y\) is \(H(s) = C(sI_n - A)^{-1}B\), which is a strictly minimum phase and has relative degree \(r \in \mathbb{N}\); then system (A7) admits state-space representation (70) in [23], which satisfies Assumption 10 of [23], under some real invertible coordinate transformation, with index \(r_1 = r\).

**Proof**
First, we introduce a linear state diffeomorphism for (A7) to decompose the system into observable and unobservable part. We choose a real invertible matrix \(T\) to decompose the system into observable and unobservable parts. The equivalence transformation \(x = T\tilde{x}\) will transform the dynamics of \(x\) into
\[
\begin{bmatrix}
\dot{\tilde{x}}_1 \\
\dot{\tilde{x}}_2 \\
\dot{\tilde{x}}_3 \\
\tilde{z}_o
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{12,3} & 0 \\
A_{21} & A_{22} & 0 & 0 \\
0 & 0 & A_{\bar{o},12} & A_{\bar{o},3} \\
A_{\bar{o},1} & A_{\bar{o},13} & A_{\bar{o},\bar{r}}
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{z}_o
\end{bmatrix}
\]

\[
y = [C_1 \ 0 \ C_3 \ 0][\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3 \ \tilde{z}_o] + Ew
\]

where \(z_o\) is an \(n\)-dimensional vector, \(n \in \mathbb{N}\) and \(\bar{o}\) is Hurwitz.

Now consider the \(z_o\) dynamics. By Lemma 13 of [23], there exists a real invertible matrix \(T\), such that the state transformation \(z_o = T[x_1' \ x_2' \ x_3' \ x_4']'\) brings system (A8) into the following state-space representation. Here, we note that \(x_4\) is actually empty since the matrix \(A\) is Hurwitz:

\[
\begin{align*}
\dot{\tilde{x}}_1 &= A_{11} \tilde{x}_1 + A_{12} \tilde{x}_2 + A_{12,3} \tilde{x}_3 + B_1 u + D_1 w \\
\dot{\tilde{x}}_2 &= A_{21} \tilde{x}_1 + A_{22} \tilde{x}_2 + D_2 w \\
\dot{\tilde{x}}_3 &= A_{\bar{o},12} \tilde{x}_1 + A_{\bar{o},13} \tilde{x}_2 + A_{\bar{o},\bar{r}} \tilde{z}_o + D_3 w \\
y &= [C_1 \ 0 \ C_3 \ 0][\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3 \ \tilde{z}_o] + Ew
\end{align*}
\]

where the partitions are compatible with the partition of the state vector; the matrices \(A_{11}, A_{12}, A_{21}, A_{22}, B_1\), and \(C_1\) admit the following structures:

\[
A_{11} = \begin{bmatrix}
a_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_{r-1} \\
a_r & \cdots & \underbrace{0_{1 \times (r-1)}}_{\bar{a}_r}
\end{bmatrix}
\]

\[
A_{12} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
\bar{a}_r
\end{bmatrix}
\]

\[
A_{21} = \begin{bmatrix}
0_{n_1 \times (r-1)} \\
\vdots \\
0_{n_{\bar{o}_1} \times (r-1)}
\end{bmatrix}
\]
By Lemma A.2, W.L.O.G., assume 

\[ \dot{n}_1 - r \geq 1 \]

admits state space representation (70) as in [23], which satisfies Assumption 10 of [23] with \( r_1 = r_2 \), under some real invertible coordinate transformation.

**Proof**

By analysis that is similar to that in the Case 2 in the proof of Lemma 3, there exists a real invertible matrix \( T_1 \) such that system (1) admits the state-space representation (A5) in the coordinates of \( T_1^{-1} \dot{x} = [x_c' \ z_1' \ z_2'] \), where \( x_c \) is \( n_1 \)-dimensional, \( n_1 \in \mathbb{N} \cup \{0\} \); the triple \((A_c, B_c, C_c)\) is controllable and observable; the matrix \( A_{33} \) is Hurwitz; all of the eigenvalue of the matrix \( \dot{A}_{44} \) are on the joω-axis and all Jordan blocks of \( \dot{A}_{44} \) are of order 1.

Now, we will discuss two exhaustive and mutually exclusive cases: Case 1: \( n_1 > 0 \); Case 2: \( n_1 = 0 \). First, consider Case 1: \( n_1 > 0 \). The transfer function of (A5) from \( u \) to \( y \) is

\[ H(s) = C_c (sI_{n_1} - A_c)^{-1} B_c + b_0 \]

where \( a_i, \ i = 1, \ldots, n_1, b_j, \ j = 0, \ldots, n_1, \) are some constants and \( b_0 \neq 0 \). Then, there exists a \( r_0 \in \{1, \ldots, n_1\} \), such that \( b_{r_0} \neq 0 \), and \( b_j = 0, 1 \leq j \leq r_0 - 1 \).

By Lemma 12 of [23], there exists a real invertible matrix \( T \), which transforms the triple \((A_c, B_c, C_c)\) into

\[
\begin{bmatrix}
T^{-1} & 0_{n_1 \times 1} \\
0_{1 \times n_1} & 1 \\
\end{bmatrix}
\begin{bmatrix}
A_c & B_c \\
C_c & 0 \\
\end{bmatrix}
\begin{bmatrix}
T \\
0_{1 \times n_1} \\
\end{bmatrix}
\begin{bmatrix}
0_{n_1 \times 1} \\
1 \\
\end{bmatrix}
\]

where

\[
\tilde{A}_{11} = \begin{bmatrix}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_{r_0-1} \\
\tilde{a}_{r_0} \\
\end{bmatrix}
\begin{bmatrix}
I_{r_0-1} \\
0_{1 \times (r_0-1)} \\
\end{bmatrix}
\]

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Then, the coordinate transformation $T_3^{-1}T_4^{-1}[\eta', \dot{x}'] = [\eta', \ddot{x}_1', \ddot{x}_2', \ddot{x}_3', \ddot{x}_4']$, where $T_3 = \text{block diagonal} \left( I_{n_2}, T, I_{n-n_1} \right)$, $T_4 = \text{block diagonal} \left( I_{n_2}, T_1 \right)$, will transform system (A12) into

$$
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 \\
\ddot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-A_{11} & A_{12} & A_{13} & A_{14} \\
0 & -A_{21} & A_{22} & A_{23} & A_{24} \\
0 & 0 & 0 & \dot{A}_{33} \\
0 & 0 & 0 & 0 & \dot{A}_{44}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
+ \frac{B_2}{b_0} \begin{bmatrix}
B_2 \dot{C}_1 \\
B_2 \dot{C}_2 \\
B_2 \dot{C}_3 \\
B_2 \dot{C}_4
\end{bmatrix} + \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
$$

We will focus on the $[\eta', \ddot{x}_1', \ddot{x}_2', \ddot{x}_3']'$ dynamics. First, we claim that any uncontrollable mode of the following pair

$$
\begin{bmatrix}
\lambda I - A_2 & -B_2 \dot{C}_1 & 0 \\
0 & \lambda I - \dot{A}_{11} & \ddot{A}_{12} \\
0 & -\ddot{A}_{21} & \lambda I - \ddot{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{b}_0 \dot{B}_2 \\
\dot{B}_1
\end{bmatrix}
$$

is an eigenvalue of the matrix $A_2$, which has negative real part. We show it as follows. Let $\lambda \in \mathbb{C}$ be a uncontrollable mode of the above pair, then the matrix

$$
\begin{bmatrix}
\lambda I - \ddot{A}_{11} & -\ddot{A}_{12} \\
-\ddot{A}_{21} & \lambda I - \ddot{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{b}_0 \dot{B}_2 \\
\dot{B}_1
\end{bmatrix}
$$

has full row rank for any $\lambda$, it implies that the matrix $\lambda I - A_2$ is singular, i.e. $\lambda$ is an eigenvalue of $A_2$.

Next, we claim that any unobservable mode of the following pair:

$$
\begin{bmatrix}
\lambda I - A_2 & -B_2 \dot{C}_1 & 0 \\
0 & \lambda I - \ddot{A}_{11} & \ddot{A}_{12} \\
0 & -\ddot{A}_{21} & \lambda I - \ddot{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{b}_0 \dot{B}_2 \\
\dot{B}_1
\end{bmatrix}
$$

is an eigenvalue of $A_2$ or is a zero of $H_2(s)$, which then has negative real part. We will prove it as follows. Let $\lambda \in \mathbb{C}$ be an unobservable mode, then the following matrix

$$
\begin{bmatrix}
\dot{\eta} \\
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
A_2 & B_2 \dot{C}_1 & 0 & B_2 \dot{C}_3 & B_2 \dot{C}_4 \\
0 & A_{11} & A_{12} & A_{13} & A_{14} \\
0 & A_{21} & A_{22} & A_{23} & A_{24} \\
0 & 0 & 0 & \dot{A}_{33} & 0 \\
0 & 0 & 0 & 0 & \dot{A}_{44}
\end{bmatrix}
\begin{bmatrix}
\eta \\
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
+ \begin{bmatrix}
B_2 b_0 \\
B_1 \\
0 \\
0
\end{bmatrix} \dot{u} + \begin{bmatrix}
B_2 \dot{E} + D_2 & 0 & 0
\end{bmatrix} \dot{w}
$$

(A13a)

$$
\eta_L = [C_2 | 0 | 0 | 0 | 0] \dot{\eta} + \dot{E}_2 \dot{w}
$$

(A13b)
must be singular, i.e.

\[
\det \begin{bmatrix}
C_2 & 0 \\
\lambda I - A_2 & -B_2
\end{bmatrix}
= 0 = \det \begin{bmatrix}
0 & C_2 \\
-B_2 & \lambda I - A_2
\end{bmatrix}
\]

There are two exhaustive and mutually exclusive cases. Case 1a: the matrix \(\lambda I - A_2\) is singular, it implies the unobservable mode is an eigenvalue of \(A_2\). Case 1b: the matrix \(\lambda I - A_2\) is invertible. Then the above determinant equality implies

\[
\det \begin{bmatrix}
C_2(\lambda I - A_2)^{-1} B_2 & 0 \\
0 & \lambda I - A_2
\end{bmatrix}
= H_2(\lambda) \det(\lambda I - A_2) = 0
\]

This further implies that \(\lambda\) is a zero of \(H_2(\lambda)\). Therefore, this claim is proved.

Next, we decompose the \([\eta' \ xco_1' \ xco_2']'\) dynamics into observable and unobservable parts, and then decompose the observable part into controllable and uncontrollable part. The real invertible transformation \([x_{co} \ x_{co}' \ x_{o}]' = T_{co}^{-1} [\eta' \ xco_1' \ xco_2']'\) will transform (A13) into

\[
\begin{bmatrix}
\dot{x}_{co} \\
\dot{x}_{\tilde{co}} \\
\dot{x}_{\tilde{o}} \\
\dot{\tilde{x}}_3 \\
\dot{\tilde{x}}_4
\end{bmatrix}
= \begin{bmatrix}
A_{co} & * & 0 & * & * \\
0 & A_{\tilde{co}} & 0 & * & * \\
* & * & A_{\tilde{o}} & * & * \\
0 & 0 & \tilde{A}_{33} & 0 & \tilde{x}_3 \\
0 & 0 & 0 & \tilde{A}_{44} & \tilde{x}_4
\end{bmatrix}
\begin{bmatrix}
x_{co} \\
x_{\tilde{co}} \\
x_{\tilde{o}} \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix}
+ \begin{bmatrix}
B_{co} \\
0 \\
0 \\
0 \\
0
\end{bmatrix}u + \begin{bmatrix}
D_{co} \\
D_{\tilde{co}} \\
D_{\tilde{o}} \\
0 \\
0
\end{bmatrix}\dot{w} \tag{A15}
\]

\[
\eta_L = [C_{co} \ C_{\tilde{co}} \ 0 \ 0 \ 0][x_{co}' \ x_{co}' \ x_{o}' \ xco_1' \ xco_2']' + E_2\dot{w} \tag{A16a}
\]

\[
\eta_L = [C_{11} \ 0 \ C_{\tilde{co}} \ 0 \ 0][x_{co}' \ x_{co}' \ x_{o}' \ xco_1' \ xco_2']' + E_2\dot{w} \tag{A16b}
\]
where *'s stand for some arbitrary constant matrices;

\[
\begin{align*}
A_{111} &= \begin{bmatrix}
\tilde{a}_1 & 1_{r_2-1} \\
\vdots & \vdots \\
\tilde{a}_{r_2-1} & 1_{1 \times (r_2-1)} \\
\tilde{a}_{r_2} & 0_{1 \times (r_2-1)}
\end{bmatrix} \\
A_{112} &= \begin{cases}
\begin{bmatrix}
0 \\
1 \\
\vdots \\
0_{n_3 \times (n_3-r_2)} \\
\vdots \\
0_{n_3 \times (n_3-r_2)} \\
\vdots \\
0_{n_3 \times (n_3-r_2)}
\end{bmatrix} & n_3 > r_2 \\
\begin{bmatrix}
1 \\
\vdots \\
1 \\
\vdots \\
1
\end{bmatrix} & n_3 = r_2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
A_{121} &= \begin{bmatrix}
\tilde{a}_{r_2+1} \\
\vdots \\
\tilde{a}_{n_3}
\end{bmatrix} \\
A_{122} &= \begin{cases}
\begin{bmatrix}
-\tilde{b}_{r_2+1} & \tilde{b}_{r_2} \\
-\tilde{b}_{n_3-1} & \tilde{b}_{n_3}
\end{bmatrix} & n_3 > r_2 \\
\begin{bmatrix}
0_{1 \times (n_3-r_2)} \\
\vdots \\
0_{1 \times (n_3-r_2)} \\
\vdots \\
0_{1 \times (n_3-r_2)} \\
\vdots \\
0_{1 \times (n_3-r_2)} \\
\vdots \\
0_{1 \times (n_3-r_2)}
\end{bmatrix} & n_3 = r_2
\end{cases}
\end{align*}
\]

\[
B_{11} = [0_{1 \times (r_2-1)} \; \tilde{b}_{r_2}] \; 
C_{11} = [1 \; 0_{1 \times (r_2-1)}]
\]

where \( \tilde{a}_i, \; i = 1, \ldots, n_3, \tilde{b}_i, \; i = r_2, \ldots, n_3, \) are some constants; \( \tilde{b}_{r_2} \neq 0; \) the matrix \( A_{122} \) is Hurwitz, and the dimension of \( \tilde{x}_1 \) is \( r_2. \) We choose \( \tilde{x}_3 = [x'_c \; \tilde{x}_3']' \), \( \tilde{x}_4 = \tilde{x}_4, \) and \( \tilde{x}_5 = \tilde{x}_5. \) Then the composite system (A16), in the coordinate of \( [\tilde{x}'_1 \; \tilde{x}'_2 \; \tilde{x}'_3 \; \tilde{x}'_4 \; \tilde{x}'_5]' \), admits the state-space representation (70) as in [23] and satisfies Assumption 10 of [23] with \( r_1 = r_2. \)

Case 2, \( n_1 = 0, \) can be proven by a line of reasoning that is similar to that of case 1.

This completes the proof of this lemma. \( \square \)

REFERENCES


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